

CURVATURE AND THE SECOND FUNDAMENTAL FORM IN CLASSIFYING QUASI-HOMOGENEOUS HOLOMORPHIC CURVES AND OPERATORS IN THE COWEN-DOUGLAS CLASS

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ABSTRACT. In this paper we study quasi-homogeneous operators, which include the homogeneous operators, in the Cowen-Douglas class. We give two separate theorems describing canonical models (with respect to equivalence under unitary and invertible operators, respectively) for these operators using techniques from complex geometry. This considerably extends the similarity and unitary classification of homogeneous operators in the Cowen-Douglas class obtained recently by the last author and A. Korányi. Specifically, the complex geometric invariants used for our classification are the curvature and the second fundamental forms inherent in the definition of a quasi-homogeneous operator. We show that these operators are irreducible and determine when they are strongly irreducible. Applications include the equality of the topological and algebraic K -group of a quasi-homogeneous operator and an affirmative answer to a well-known question of Halmos on similarity for these operators.

1. INTRODUCTION

Let \mathcal{H} be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on \mathcal{H} . For an open connected subset Ω of the complex plane \mathbb{C} , and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in their very influential paper [2]. An operator T acting on a Hilbert space \mathcal{H} belongs to this class if each $w \in \Omega$, is an eigenvalue of the operator T of constant multiplicity n , these eigenvectors span the Hilbert space \mathcal{H} and the operator $T - w$, $w \in \Omega$, is surjective. They showed that for an operator T in $B_n(\Omega)$, there exists a holomorphic choice of n linearly independent eigenvectors, that is, the map $w \rightarrow \ker(T - w)$ is holomorphic. Thus $\pi : E_T \rightarrow \Omega$, where

$$E_T = \{\ker(T - w) : w \in \Omega, \pi(\ker(T - w)) = w\}$$

defines a Hermitian holomorphic vector bundle on Ω .

We recall some of the basic definitions from [2] before stating one of its main results. The Grassmanian $\text{Gr}(n, \mathcal{H})$, is the set of all n -dimensional subspaces of the Hilbert space \mathcal{H} . A map $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ is said to be a holomorphic curve, if there exist n (point-wise linearly independent) holomorphic functions $\gamma_1, \gamma_2, \dots, \gamma_n$ on Ω taking values in a Hilbert space \mathcal{H} such that $t(w) = \vee\{\gamma_1(w), \dots, \gamma_n(w)\}$, $w \in \Omega$. Any holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ gives rise to a n -dimensional Hermitian holomorphic vector bundle E_t over Ω , namely,

$$E_t = \{(x, w) \in \mathcal{H} \times \Omega \mid x \in t(w)\} \text{ and } \pi : E_t \rightarrow \Omega, \text{ where } \pi(x, w) = w.$$

Given two holomorphic curves $t, \tilde{t} : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$, if there exists a unitary operator U on \mathcal{H} such that $\tilde{t} = Ut$, that is, the restriction $U(w) := U|_{E_t(w)}$ of the unitary operator U to the fiber $E_t(w)$ of E at w maps it to the fiber of $E_{\tilde{t}}(w)$, then t and \tilde{t} are said to be congruent. If t and \tilde{t} are congruent, then clearly the vector bundles E_t and $E_{\tilde{t}}$ are equivalent via the holomorphic bundle map induced by the unitary operator U . Furthermore, t and \tilde{t} are said to be similar if there exists an invertible

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operator $X \in \mathcal{L}(H)$ such that $\tilde{t} = Xt$, that is, $X(w) := X|_{E_t(w)}$ is an isomorphism except that $X(w)$ is no longer an isometry. In this case, we say that the vector bundles E_t and $E_{\tilde{t}}$ are similar.

An operator T in the class $B_n(\Omega)$ determines a holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$, namely, $t(w) = \ker(T - w)$, $w \in \Omega$. However, if t is a holomorphic curve, setting $Tt(w) = wt(w)$, defines a linear transformation on a dense subspace of the Hilbert space \mathcal{H} . In general, we have to impose additional conditions to ensure that the operator T is bounded. Assuming that t defines a bounded linear operator T , unitary and similarity invariants for the operator T are then obtained from those of the vector bundle E_t . To describe these invariants, we need the curvature of the vector bundle E_t along with its covariant derivatives. Let us recall some of these notions following [2].

The Hermitian structure of the holomorphic bundle E_t , with respect to a holomorphic frame γ is given by the Grammian

$$h_\gamma(w) = \left(\langle \gamma_j(w), \gamma_i(w) \rangle \right)_{i,j=1}^n, \quad w \in \Omega.$$

If we let $\bar{\partial}$ denote the complex structure of the vector bundle E_t , then the connection compatible with both the complex structure ∂ and the metric h is canonically determined and is given by the formula $h^{-1}\partial h dz$. The curvature of the holomorphic Hermitian tor bundle E_f is then the $(1, 1)$ form

$$K(w) = -\partial(h_\gamma^{-1}\bar{\partial}h_\gamma)dw \wedge d\bar{w}.$$

We let $\mathcal{K}(w)$ denote the coefficient of this $(1, 1)$ form, that is, $\mathcal{K}(w) := -\frac{\partial}{\partial \bar{w}}(h_\gamma^{-1}(w)\frac{\partial}{\partial w}h_\gamma(w))$. Thus it is an endomorphism of the fiber $E_t(w)$.

Let E be a C^∞ vector bundle with a Hermitian structure. We are not assumed to be holomorphic. The derivatives of a bundle map $\phi : E \rightarrow E$ with respect to a frame γ is defined to be

- (1) $(\phi_\gamma)_{\bar{w}} = \frac{\partial}{\partial \bar{w}}(\phi_\gamma)$;
- (2) $(\phi_\gamma)_w = \frac{\partial}{\partial w}(\phi_\gamma) + [h_\gamma^{-1}\frac{\partial}{\partial w}h_\gamma, \phi_\gamma], w \in \Omega$.

Since the curvature \mathcal{K} may be thought of as a bundle map, its partial derivatives $\mathcal{K}_{w^{i\bar{w}^j}}$, $i, j \in \mathbb{N} \cup \{0\}$, may be defined inductively. The curvature and its derivatives are unitarily invariants of the holomorphic Hermitian vector bundle E_t , what is more, a finite subset of these form a complete set of invariants as was shown in [2]. For this and other deep connections between operator theory and complex geometry, we refer the reader to [2].

Theorem (Proposition 2.8, [2]). *Two holomorphic Hermitian bundles E_t and $E_{\tilde{t}}$ are equivalent if and only if there exists an isometric (holomorphic) bundle map $V : E_t \rightarrow E_{\tilde{t}}$ such that*

$$V((\mathcal{K}_t)_{w^{i\bar{w}^j}}) = ((\mathcal{K}_{\tilde{t}})_{w^{i\bar{w}^j}})V, \quad i, j = 0, 1, \dots, n-1.$$

It was observed in [2] that the local nature of the Complex geometric invariants limits their use in the study of the equivalence under an invertible linear transformation. The global nature of such an equivalence is not easily detected by local invariants like the curvature and its derivatives. However, many interesting results were obtained in [2] involving the question of similarity. In the absence of a characterization of the equivalence classes under an invertible linear transformation, a conjecture was made for two operators in $B_1(\mathbb{D})$ to be similar. Unfortunately, this conjecture turned out to be false (cf. [3, 4]). More recently, Jiang and Ji obtained the following result on similarity, which is best described in terms of the commutant

$$\mathcal{A}'(t \oplus \tilde{t}) = \{T \in \mathcal{L}(\mathcal{H}) | T(t(w) \oplus \tilde{t}(w)) \subseteq t(w) \oplus \tilde{t}(w), \quad w \in \Omega\}$$

of two holomorphic curves t and \tilde{t} .

Theorem (Theorem 3.1, [17]). *Suppose $t, \tilde{t} : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ are two holomorphic curves. Then the ordered K_0 group of $\mathcal{A}'(t \oplus \tilde{t})$ is a complete similarity invariant of t and \tilde{t} .*

Describing similarity invariants in terms of the curvature and its derivatives has been somewhat more elusive except for the very recent results of R. G. Douglas, H. Kwon and S. Treil [25, 7].

The motivation for this work comes from three very different directions. The attempt is to describe a canonical model and obtain invariants for operators in the Cowen-Douglas class with respect to equivalence under conjugation under a unitary or invertible linear transformation. These questions have been successfully addressed using ideas from K -theory and representation theory of Lie groups.

First, the detailed study of the Cowen-Douglas class of operators, reported in the book [19], begins with the following basic structure theorem for these operators.

Theorem (Theorem 1.49, [19]). *If T is an operator in the Cowen-Douglas class $B_n(\Omega)$, then there exists operators T_0, T_1, \dots, T_{n-1} in $B_1(\Omega)$ such that*

$$(1.1) \quad T = \begin{pmatrix} T_0 & S_{0,1} & * & \cdots & * \\ 0 & T_1 & S_{1,2} & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}.$$

A slight paraphrasing clearly implies that if $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ is a holomorphic frame for the vector bundle E_t , and $\mathcal{H} = \bigvee \{\gamma_i(w), w \in \Omega, 0 \leq i \leq n-1\}$, then there exists non-vanishing holomorphic curves $t_i : \Omega \rightarrow \text{Gr}(1, \mathcal{H}_i)$, $0 \leq i \leq n-1$, such that

$$(1.2) \quad \gamma_j = \phi_{0,j}(t_0) + \cdots + \phi_{i,j}(t_i) + \cdots + \phi_{j-1,j}(t_{j-1}) + t_j, \quad 0 \leq j \leq n-1,$$

where $\phi_{i,j}$ are certain holomorphic bundle maps. One would expect these bundle maps to reflect the properties of the operator T . However the tenuous relationship between the operator T and the bundle maps $\phi_{i,j}$ becomes a little more transparent *only* after we impose a natural set of constraints.

Here we have chosen not to distinguish between the holomorphic curve t of rank 1 in the projective space $\text{Gr}(1, \mathcal{H})$ and a non-vanishing section of the line bundle E_t , that is, we have let t represent them both. This will be our convention through out the paper.

Secondly, to a large extent, these constraints were anticipated in the recent paper [15, 16]. In that paper, a class of operators $\mathcal{F}B_n(\Omega)$ in $B_n(\Omega)$ possessing, what we called, a flag structure were isolated. The flag structure was shown to be rigid. It was then shown that the complex geometric invariants like the curvature and the second fundamental form of the vector bundle E_T are indeed unitary invariants of the operator T . However, to show that these form a complete set of unitary invariants, we had to impose additional constraints and introduce an even smaller class $\tilde{\mathcal{F}}B_n(\Omega)$. For the operators $\mathcal{F}B_n(\Omega)$, it turned out that the bundle maps $\phi_{j,j+1}$, $0 \leq j \leq n-1$, were constant while for those in the smaller class $\tilde{\mathcal{F}}B_n(\Omega)$, the remaining maps $\phi_{j,k}$ were all zero.

Finally, recall that an operators T in $B_n(\mathbb{D})$ is said to be homogeneous if the unitary orbit of T under the action of the Möbius group is itself, that is, $\varphi(T)$ is unitarily equivalent to T for φ in some open neighbourhood of the identity in the Möbius group (cf. [1]). A canonical element $T^{(\lambda, \mu)}$ in each unitary equivalence class of the homogeneous operators in $B_n(\mathbb{D})$ was constructed in [23]. It was then shown that two operators $T^{(\lambda, \mu)}$ and $T^{(\lambda', \mu')}$ are similar if and only if $\lambda = \lambda'$. In particular choosing $\mu = 0$, one verifies that a homogeneous operator in $B_n(\mathbb{D})$ is similar to the n -fold direct sum $T_0 \oplus \cdots \oplus T_n$, where T_i is the adjoint of the multiplication operator $M^{(\lambda_i)}$ acting on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$ determined by the positive definite kernel $\frac{1}{(1-z\bar{w})^{\lambda_i}}$ defined the unit disc \mathbb{D} , $0 \leq i \leq n-1$, $\lambda_i > 0$.

The homogeneous operators are easily seen to be in the class $\mathcal{F}B_n(\mathbb{D})$ and the operator corresponding to the Hilbert module \mathcal{M}_{loc} is in $\tilde{\mathcal{F}}B_n(\Omega)$, (cf. [16])

In this paper we study a class of operators, to be called quasi-homogeneous, for which we can prove results very similar to those for the homogeneous operators building on the techniques developed in [16]. This class of operators, as one may expect, contains the homogeneous operators and is characterized by the requirement that all the bundle maps of (1.2) take their values in a certain (full) jet bundle $\mathcal{J}_i(t)$ of the holomorphic curve t . For a detailed account of the jet bundles, we refer the reader to [29].

Definition 1.1. *If t is a holomorphic curve in the Grassmannian of rank 1, that is, $t : \Omega \rightarrow \text{Gr}(1, \mathcal{H})$. Let $\gamma(w)$ be a non-vanishing holomorphic section for the line bundle E_t . The derivatives $\gamma^{(j)}$, $j \in \mathbb{N}$, taking values again in the Hilbert space \mathcal{H} are holomorphic. (It can be shown that they are linearly independent.) The jet bundle $\mathcal{J}_n E_t(\gamma)$ is defined by the holomorphic frame $\{\gamma^{(0)}(w) := \gamma, \gamma^{(1)}, \dots, \gamma^{(n)}\}$. Since t is a holomorphic curve, the vectors $\gamma^{(i)}(w)$ and $\gamma^{(j)}(w)$ are in the Hilbert space \mathcal{H} . Therefore the inner product of these two vectors is defined using that of the Hilbert space \mathcal{H} .*

In the following definition we assume, implicitly, that the bundle map $\phi_{i,j}$ of (1.2) are from the holomorphic line bundles E_i to a jet bundle $\mathcal{J}_j E_i$, where for brevity of notation and when there is no possibility of confusion, we will let E_i denote the vector bundle induced by the holomorphic curve t_i , $0 \leq i \leq n-1$.

Definition 1.2. *Let t be a holomorphic curve with a holomorphic frame $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ in the Grassmannian $\text{Gr}(n, \mathcal{H})$ of a complex separable Hilbert space \mathcal{H} . We say that t has an atomic decomposition if there exists holomorphic curves $t_i : \Omega \rightarrow \text{Gr}(1, \mathcal{H}_i)$, to be called the atoms of t , corresponding to operators $T_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ in $B_1(\mathbb{D})$ and complex numbers $\mu_{i,j} \in \mathbb{C}$, $0 \leq j \leq i \leq n-1$, such that $\mathcal{H} = \mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_{n-1}$ and*

$$\begin{aligned} \gamma_0 &= \mu_{0,0} t_0 \\ \gamma_1 &= \mu_{0,1} t_0^{(1)} + \mu_{1,1} t_1 \\ \gamma_2 &= \mu_{0,2} t_0^{(2)} + \mu_{1,2} t_1^{(1)} + \mu_{2,2} t_2 \\ &\vdots \\ \gamma_j &= \mu_{0,j} t_0^{(j)} + \dots + \mu_{i,j} t_i^{(j-i)} + \dots + \mu_{j,j} t_j \\ &\vdots \\ \gamma_{n-1} &= \mu_{0,n-1} t_0^{(n-1)} + \dots + \mu_{i,n-1} t_i^{(n-1-i)} + \dots + \mu_{n-1,n-1} t_{n-1}. \end{aligned}$$

Fix i in $\{0, \dots, n-1\}$. We say that the holomorphic curve t_i is homogeneous if for $w \in \mathbb{D}$, $\mathbb{C}[t_i(w)] = \ker(T_i - w)$ for some homogeneous operator T_i in $B_1(\mathbb{D})$. We realize, up to unitary equivalence, such a homogeneous operator T_i in $B_1(\mathbb{D})$ as the adjoint of the multiplication operator $M^{(\lambda_i)}$ on the weighted Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. Thus for a fixed $w \in \mathbb{D}$, there exists a canonical (holomorphic) choice of eigenvectors $t_i(w)$, namely, $(1 - z\bar{w})^{-\lambda_i}$.

We say that t is quasi-homogeneous if it admits an atomic decomposition, where each of the atoms t_i is homogeneous, $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$ and the difference $\lambda_{i+1} - \lambda_i$, $0 \leq i \leq n-2$, is a fixed positive real number, say, $\Lambda(t)$.

When the holomorphic curve defines a bounded linear operator, we shall use the terms quasi-homogeneous holomorphic curve t , quasi-homogeneous operator T and quasi-homogeneous holomorphic vector bundle E_t (or, even E_T) interchangeably.

If T is a quasi-homogeneous operator and $(S_{i,j})$ is its upper triangular decomposition given in Theorem 1, then we show that

$$(1.3) \quad T_i S_{i,i+1} = S_{i,i+1} T_{i+1}, \quad 0 \leq i \leq n-2.$$

In consequence, all quasi-homogeneous operators belong to the class $\mathcal{FB}_n(\mathbb{D})$ introduced recently in the paper [15, 16].

One of the points of this definition is that a quasi-homogeneous vector bundle E_t is indeed homogeneous if $\Lambda(t) = 2$ and the constants $\mu_{i,j}$ are certain explicit functions of λ as we point out at the end of the following section. However, a quasi-homogeneous vector bundle need not be homogeneous as the following example shows.

Example 1.3. Let S be the adjoint of the multiplication operator on arbitrary weighted Bergmann space $\mathbb{A}^{(\lambda)}(\mathbb{D})$ and let T be the operator

$$T = \begin{pmatrix} S & \mu_1 I & 0 & \cdots & 0 \\ 0 & S & \mu_2 I & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S & \mu_n I \\ 0 & \cdots & \cdots & 0 & S \end{pmatrix}, \mu_i \in \mathbb{C},$$

defined on the $n + 1$ fold direct sum $\bigoplus_{i=1}^{n+1} \mathbb{A}^{(\lambda)}(\mathbb{D})$. Then T is in $\tilde{\mathcal{F}}B_{n+1}(\mathbb{D})$ and therefore belongs to $B_{n+1}(\mathbb{D})$ and the corresponding holomorphic curve $t(w) = \ker(T - w)$, $w \in \mathbb{D}$, is quasi homogeneous with $\Lambda(t) = 0$. In fact, in this Example, if we replace S with an arbitrary operator, say R , from $B_1(\mathbb{D})$, then the resulting operator T while no longer quasi-homogeneous, remains a member of $\tilde{\mathcal{F}}B_{n+1}(\mathbb{D})$. Indeed, it has already appeared, via module tensor products, in our earlier work [16, Section 3.1].

The class of quasi-homogeneous operators, contrary to what might appear to be a rather small class of operators, contains apart from the homogeneous operators, many other operators. Indeed, in rank 2, for instance, it is parametrized by the multiplier algebra of two homogeneous operators. In the definition of the quasi-homogeneous operators given above, if we let the atoms occur with some multiplicity rather than being multiplicity-free, it will make it even larger. This would cause additional complications, which we are not able to resolve at this time. In another direction, we need not assume that the atoms themselves are homogeneous. Most of our results would appear to go through if we merely assume that the kernel function $K^{(\lambda)}(w, w) \sim \frac{1}{(1-|w|^2)^\lambda}$, $|w| < 1$. Deep results about such functions were obtained by Hardy and Littlewood (cf. [12]) and have already appeared in the context of similarity, see [4].

A Hermitian holomorphic bundle E is said to be irreducible, if E can not be written as orthogonal direct sum of two holomorphic sub-bundles of E . In the paper [2], among other things, it is shown that an operator T on $B_n(\Omega)$ is irreducible if and only if the holomorphic Hermitian vector bundle E_T is irreducible.

Let $X : \mathcal{H} \rightarrow \mathcal{H}$ be an invertible bounded linear operator and let XE_T be the holomorphic Hermitian vector bundle obtained by prescribing the fiber at $w \in \Omega$ to be $X(E_T(w))$. A Hermitian holomorphic bundle E_T is said to be strongly irreducible, if XE_T cannot be written as orthogonal direct sum of two holomorphic sub-bundles for any invertible linear operator X , again, the vector bundle E_T is strongly irreducible if and only if the operator T is strongly irreducible (cf. [17, 19, 20]). It was proved in [17] that a holomorphic curve is strongly irreducible if and only if there is no non-trivial idempotent in the commutant $\mathcal{A}'(t)$. We determine which of the quasi-homogeneous operators is strongly irreducible and use this information to give a canonical model for the equivalence (under unitary as well as invertible transformations) class of quasi-homogeneous operators. We recall one more notion from complex geometry which will be necessary to describe the main results of this paper.

If E is a holomorphic Hermitian vector bundle and E_0 is a holomorphic sub-bundle of E , then $E = E_0 \oplus E_0^\perp$, where $E_0^\perp(w)$ is orthogonal complement of $E_0(w)$, $w \in \Omega$. However, as is well-known [21], this decomposition is holomorphic if and only if the second fundamental form of E_0 in E is zero. The quasi-homogeneous holomorphic Hermitian vector bundles admit an atomic decomposition and each of these atoms are holomorphic line bundles each of which is nested in the next one via a bundle map. Therefore, it turns out, the second fundamental form of a neighboring pair of atoms is all that matters. Fortunately, this can be explicitly described as follows. The 2×2 block $\begin{pmatrix} S_{i,i} & S_{i,i+1} \\ 0 & S_{i+1,i+1} \end{pmatrix}$ in the decomposition of the operator T given in Theorem 1 is in $\mathcal{F}B_2(\mathbb{D})$ because of the intertwining property (1.3). Hence the corresponding second fundamental form θ_i of the inclusion E_{γ_i} in $E_{\{\gamma_i, \gamma_{i+1}\}}$ (cf. [16, Section 2.5] and [9, Section 5.1]) is given by the formula

$$(1.4) \quad \theta_i(z) = \frac{\mu_{i,i+1} \mathcal{K}_i(z) d\bar{z}}{\left(\frac{\|t_{i+1}(z)\|^2}{\|t_i(z)\|^2} - |\mu_{i,i+1}|^2 \mathcal{K}_i(z) \right)^{1/2}}.$$

We now describe, without going in to too many details, the main results of this paper. Fix a quasi-homogeneous operator T (respectively \tilde{T}), or equivalently, a holomorphic curve t (respectively \tilde{t}).

- (1) A quasi-homogeneous operator admits an upper triangular representation in terms of its atoms and it belongs to the class $\mathcal{FB}_n(\mathbb{D})$ introduced recently in [16].
- (2) A quasi-homogeneous operator, or equivalently, a quasi-homogeneous holomorphic vector bundle E_T is irreducible;
- (3) If the operator T is quasi-homogeneous and $\Lambda(t) < 2$, then T is strongly irreducible, and if $\Lambda(t) \geq 2$ then T is strongly reducible.
- (4) If t and \tilde{t} are quasi-homogeneous holomorphic curves, which are unitarily equivalent, then we have
 - (a) $\mathcal{K}_{t_i} = \mathcal{K}_{\tilde{t}_i}$, $i = 0, 1, \dots, n-1$,
 - (b) $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $i = 0, 1, \dots, n-2$, where $\theta_{i,i+1}$ (respectively, $\tilde{\theta}_{i,i+1}$) are the second fundamental forms of the inclusion E_{γ_i} in $E_{\{\gamma_i, \gamma_{i+1}\}}$ (respectively, $E_{\tilde{\gamma}_i}$ in $E_{\{\tilde{\gamma}_i, \tilde{\gamma}_{i+1}\}}$), $0 \leq i \leq n-2$.

This was proved in [16] for operators in $\mathcal{FB}_n(\Omega)$. However, here we describe a canonical element in each unitary equivalence class of a quasi-homogeneous operator and compare these canonical elements to decide if two such operators are unitarily equivalent. This appears to be a surprising rigidity property of quasi-homogeneous operators.

- (5) Assume that E_t is a quasi-homogeneous vector bundle with atoms t_i , $0 \leq i \leq n-1$. If $\Lambda(t) \geq 2$, then E_t is similar to the n -fold direct sum of the line bundles $E_{t_0}, E_{t_1}, \dots, E_{t_{n-1}}$. If $\Lambda(t) < 2$, then the description is more complicated. However, we determine exactly when two quasi-homogeneous vector bundles are similar, even in this case.
- (6) We show that the Halmos question, namely, if a bounded homomorphism of the disc algebra must be similar to a contraction, has an affirmative answer for quasi-homogeneous operators.

The paper is organized as follows. In Section 2, we describe several properties of quasi-homogeneous operators. In particular, we determine when a quasi-homogeneous holomorphic curve defines a bounded linear operator. We show that the operators appearing in the atomic decomposition of a quasi-homogeneous operator possess an important intertwining property, which is the key to much of our study. In Section 3, we find conditions which ensure two quasi-homogeneous operators are similar. It turns out that the answer depend only on the operators that appear on the diagonal and the first super diagonal of the decomposition given in Theorem 1. Section 4 contains several applications of our results on unitary equivalence and similarity of quasi-homogeneous operators. We show that if the homomorphism of the polynomial ring induced by a quasi-homogeneous operator T is bounded (for any polynomial p , there exists a constant K independent of p such that $\|p(T)\| \leq K\|p\|_{\infty, \mathbb{D}}$), then T is similar to a contraction. This gives an affirmative answer to the well-known Halmos question. We define a topological K^0 group using equivalence classes of quasi-homogeneous operators under invertible linear transformations. As a second application of our results, we show that the group K^0 is equal to the algebraic K_0 group consisting of equivalence classes of idempotents in the commutant of a quasi-homogeneous operator. In the context of the usual (topological) K^0 and (algebraic) K_0 groups, this is a consequence of the well-known theorem of R. G. Swan.

We finish this Introduction with a list of notations and conventions that we will use through out this paper.

- (a) $t : \mathbb{D} \rightarrow Gr(n, \mathcal{H})$ is a fixed but arbitrary holomorphic curve in the Grassmannian of rank n in some complex separable Hilbert space \mathcal{H} .
- (b) E_t is the rank n holomorphic Hermitian vector bundle obtained by setting $E_t(w) := t(w)$, $w \in \Omega$. If $n = 1$, we let t denote a non-vanishing holomorphic section of the line bundle E_t as well. In general, we will let $\gamma := \{\gamma_0, \dots, \gamma_{n-1}\}$ be a holomorphic frame of the vector bundle E_t .

- (c) There exists constants $\mu_{i,j} \in \mathbb{C}$, and a holomorphic frame $\gamma := \{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ of the vector bundle E_t of the form $\gamma_j = \mu_{0,j}t_0^{(k)} + \dots + \mu_{i,j}t_i^{(j-i)} + \dots + \mu_{j,j-1}t_{j-1} + t_j$, $j = 0, 1, \dots, n-1$, where $t_i : \mathbb{D} \rightarrow \text{Gr}(1, \mathcal{H}_i)$, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \cdots \oplus \mathcal{H}_{n-1}$, are holomorphic curves of rank 1 designated as the atoms of t .
- (d) T is the linear transformation defined on the dense subset $\bigvee \{t(w) : w \in \Omega\}$ by the rule $T(t(w)) = wt(w)$, $w \in \Omega$.
- (e) In this paper, we will only consider those holomorphic curves t for which the linear transformation T extends to a bounded linear operator on \mathcal{H} and is in $B_n(\mathbb{D})$. We will let E_T and E_t denote the same holomorphic Hermitian vector bundle.
- (f) A decomposition of the operator $T : \mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_{n-1} \rightarrow \mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_{n-1}$ of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}$$

is said to be *atomic* with atoms $T_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$, which are assumed to be in $B_1(\mathbb{D})$ and required to intertwine $S_{i,i+1}$, that is, $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}$, $0 \leq i \leq n-2$.

- (g) The operators $S_{i,j}$ define certain holomorphic bundle maps $s_{i,j}$ given by the rule

$$s_{i,j}(t_j(w)) = m_{i,j} t_i^{(j-i-1)}(w), \quad w \in \mathbb{D},$$

where the constants $m_{i,j}$ and $\mu_{i,j}$ determine each other recursively.

- (h) The atoms T_0, T_1, \dots, T_{n-1} of the operator T and the atoms t_0, t_1, \dots, t_{n-1} of the holomorphic curve t determine each other.
- (i) The atoms are homogeneous, that is, for $i = 0, 1, \dots, n-1$, the operator T_i is the adjoint of the multiplication operator on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}$ (cf. [26].) The weights are assumed to be increasing, the difference $\lambda_{i+1} - \lambda_i$ is assumed to be constant, say $\Lambda(t)$, which is called the *valency* of the operator T .
- (j) If T admits an atomic decomposition, the atoms are homogeneous and the valency $\Lambda(t)$ is constant, then the operator T (respectively, the holomorphic curve t and the vector bundle E_t) is said to be *quasi-homogeneous*.

In this case, the atoms T_i are assumed, without loss of generality, to have been realized as the adjoint of the multiplication operators on weighted Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$.

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2. CANONICAL MODEL UNDER UNITARY EQUIVALENCE

An operator T in the Cowen and Douglas class $B_n(\Omega)$ is determined, modulo unitary equivalence, by the curvature (of the vector bundle E_T) together with a finite number of its partial derivatives. However, if the rank n of this vector bundle is > 1 , then the computation of the curvature and its derivatives is somewhat impractical. Here we show that if the operator is quasi-homogeneous, it is enough to restrict ourselves to the computation of the curvature of the atoms and a $n-1$ second fundamental forms of pair-wise neighbouring vector bundles. We first recall, following [2, 5], that an operator T in $B_n(\Omega)$ may be realized as the adjoint of a multiplication operator on a Hilbert space of holomorphic functions on $\Omega^* := \{w : \bar{w} \in \Omega\}$ possessing a reproducing kernel.

2.1. Holomorphic Curve and Reproducing Kernel. For any operator T in $B_n(\Omega)$, let E_T denote the Hermitian holomorphic vector bundle with $E_T(w) = \ker(T - w)$, $w \in \Omega$. Let $\{\gamma_0, \dots, \gamma_{n-1}\}$ be a holomorphic frame for E_T . Define the map $\Gamma : \mathcal{H} \rightarrow \mathcal{O}(\Omega^*, \mathbb{C}^n)$ by the rule

$$\Gamma(x)(z) = (\langle x, \gamma_0(\bar{z}) \rangle, \dots, \langle x, \gamma_{n-1}(\bar{z}) \rangle)^{\text{tr}}, \quad z \in \Omega^*, \quad x \in \mathcal{H},$$

where $\mathcal{O}(\Omega^*, \mathbb{C}^n)$ is the space of holomorphic functions defined on Ω^* taking values in \mathbb{C}^n . Since the map Γ is evidently injective, we transplant the inner product from \mathcal{H} on the range of Γ , making it a Hilbert space, say \mathcal{H}_Γ . Thus Γ is now unitary by definition. Define K_Γ to be the function on $\Omega^* \times \Omega^*$ taking values in the $n \times n$ matrices $\mathcal{M}_n(\mathbb{C})$:

$$K_\Gamma(z, w) = ((\langle \gamma_j(\bar{w}), \gamma_i(\bar{z}) \rangle))_{i,j=0}^{n-1}, \quad z, w \in \Omega^*.$$

Setting $(K_\Gamma)_w(\cdot) = K_\Gamma(\cdot, w)$, we verify that

$$\langle \Gamma(x)(\cdot), (K_\Gamma)_w(\cdot) \eta \rangle_{\text{ran } \Gamma} = \langle \Gamma(x)(w), \eta \rangle_{\mathbb{C}^n}, \quad x \in \mathcal{H}, \eta \in \mathbb{C}^n, w \in \Omega^*.$$

Thus $(K_\Gamma)_w$ has the reproducing property. The unitary operator Γ intertwines the operator T with the adjoint of the multiplication operator M on the Hilbert space $(\mathcal{H}_\Gamma, K_\Gamma)$. We describe how this works for quasi-homogeneous operators. For such an operator T acting on a Hilbert space \mathcal{H} , there is a holomorphic frame $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ and atoms t_0, \dots, t_{n-1} , for which we have

$$\gamma_i = \mu_{0,i} t_0^{(i)} + \dots + \mu_{j,i} t_j^{(i-j)} + \dots + \mu_{i,i} t_i, \quad \mu_{ij} \in \mathbb{C}.$$

At this point, assuming that the operator is quasi-homogeneous makes the atoms T_0, T_1, \dots, T_{n-1} homogeneous. Conjugating with a diagonal unitary, if necessary, we assume without loss of generality that t_i is the holomorphic curve defined by

$$t_i(w) := (1 - \bar{w}z)^{-\lambda_i}, \quad \lambda_i = \lambda_0 + i \Lambda(t), \quad 0 \leq i \leq n-1, \quad \lambda_0 > 0,$$

in the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. Let $\Gamma : \mathcal{H} \rightarrow \mathcal{O}(\Omega^*, \mathbb{C}^n)$ be the map intertwining T with the adjoint of the multiplication operator on the Hilbert space of holomorphic functions on $\Omega^* = \{w : \bar{w} \in \Omega\}$ determined by the positive definite kernel

$$((K_\Gamma(z, w))_{ij} = ((\langle \gamma_j(\bar{w}), \gamma_i(\bar{z}) \rangle)), \quad z, w \in \Omega^*$$

For $1 \leq i \leq n-1$, recalling that $\Lambda(t) = \lambda_{i+1} - \lambda_i$ is a constant, the remaining atoms t_i are determined, upto a constant, say, μ_{ii} . As a consequence, setting $K_i(z, w) = \langle t_i(\bar{w}), t_i(\bar{z}) \rangle$ and $D_i = \text{diag}(0, \dots, 0, \mu_{i,i}, \mu_{i,i+1}, \dots, \mu_{i,n-1})$, $0 \leq i \leq n-1$, then we have that

$$((K_\Gamma))_{kl} = \sum_{i=0}^{n-1} D_i^* ((\partial^k \bar{\partial}^l K_i)) D_i.$$

Since the unitary equivalence class of the M^* on \mathcal{H}_Γ remains unchanged when it is conjugated by a holomorphic function, we may replace K_Γ with

$$D^{*-1} K_\Gamma D^{-1} = \sum_{i=0}^{n-1} D^{*-1} D_i^* ((\partial^l \bar{\partial}^k K_i)) D_i D^{-1},$$

where D is the $n \times n$ diagonal matrix with $\mu_{0,0}, \mu_{1,1}, \dots, \mu_{n-1,n-1}$ on its diagonal. Having done this, we assume through out this paper that $\mu_{i,i} = 1$, $0 \leq i \leq n-1$.

2.2. Atomic decomposition. Let t be a quasi-homogeneous holomorphic curve in $\text{Gr}(n, \mathcal{H})$. Assume that it defines a bounded linear operator T on the Hilbert space \mathcal{H} . An appeal to Theorem 1 provides, what we would *now* call an atomic decomposition for the operator T . This decomposition has several additional properties arising out of our assumption of quasi-homogeneity.

Lemma 2.1. *Let t be a holomorphic quasi-homogeneous curve, $\{t_0, \dots, t_{n-1}\}$ be a set of its atoms and $\{\gamma_0, \dots, \gamma_{n-1}\}$ be a holomorphic frame for E_t . Let \mathcal{H} be the closed linear span of the set of vectors $\{\gamma_0(w), \dots, \gamma_{n-1}(w) : w \in \mathbb{D}\}$ and \mathcal{H}_i be the closed linear span of the set of vectors $\{t_i(w), w \in \mathbb{D}\}$, $0 \leq i \leq n-1$. We have*

- (1) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1}$;
- (2) *There exists an operator T , defined on a dense subset of vectors in \mathcal{H}_t , which is upper triangular with respect to the direct sum decomposition $\mathcal{H}_t = \mathcal{H}_{t_0} \oplus \dots \oplus \mathcal{H}_{t_{n-1}}$:*

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \dots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \dots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & \dots & 0 & T_{n-1} \end{pmatrix},$$

where $S_{i,j}(t_j(w)) = m_{i,j}t_i^{(j-i-1)}(w)$, $T_i(t_i(w)) = wt_i(w)$, $w \in \mathbb{D}$, $i, j = 0, 1, \dots, n-1$, for some choice of complex constants $m_{i,j}$ depending on the μ_{ij} . In this case, we have $S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}$, $i = 0, 1, \dots, n-2$;

- (3) *The constants $m_{i,j}$ and $\mu_{i,j}$ determine each other.*

For convenience of notation, in the proof below, we set $S_{i,i} := T_i$, $0 \leq i \leq n-1$, in the proof. We will adopt this practice often and call T_0, T_1, \dots, T_{n-1} , the atoms of T . Also, $S_{i,i+1}(t_{i+1}) = \mu_{i,i+1}t_i$, with the assumption that $\mu_{i,i} = 1$, $0 \leq i \leq n-2$.

Proof. Note that $\{\gamma_0, \gamma_1, \dots, \gamma_{n-1}\}$ is a frame for E_t and the atoms t_i , $0 \leq i \leq n-1$ are pairwise orthogonal. From Definition 1.2, we have

$$(2.1) \quad \begin{aligned} \gamma_0 &= t_0; \\ \gamma_k &= \mu_{0,k}t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \dots + \mu_{k,k}t_k, \quad 0 \leq k < n-1; \\ \mathcal{H} &= \bigoplus_{i=0}^{n-1} \mathcal{H}_i, \quad \text{where } \mathcal{H}_i = \bigvee \{t_i(w) : w \in \Omega\}, \quad i \in \{0, 1, \dots, n-1\}. \end{aligned}$$

In particular, the first statement of the Lemma is included in the definition of a holomorphic quasi-homogeneous curve.

For $0 \leq i \leq j \leq n-1$, let $S_{i,j} : \mathcal{H}_j \rightarrow \mathcal{H}$ be the linear transformation induced by bundle maps $s_{i,j} : E_{t_j} \rightarrow \mathcal{J}_{j-i-1}E_{t_i}$, namely,

$$\sum_{i \leq j} s_{i,j}(\gamma_k(w)) = w\gamma_k(w), \quad w \in \mathbb{D}.$$

Equivalently, for any $k = 0, 1, \dots, n-1$, we have the following formulae:

$$\begin{pmatrix} s_{0,0} - w & s_{0,1} & s_{0,2} & \dots & \dots & s_{0,n-1} \\ & s_{1,1} - w & s_{1,2} & \dots & \dots & s_{1,n-1} \\ & & \ddots & & & \vdots \\ & & & s_{k-1,k-1} - w & \dots & s_{k-1,n-1} \\ & & & & \ddots & \vdots \\ & & 0 & & & s_{n-1,n-1} - w \end{pmatrix} \begin{pmatrix} \mu_{0,k}t_0^{(k)}(w) \\ \vdots \\ \mu_{k,k}t_k(w) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

It follows that

$$(2.2) \quad (s_{k,k} - w)(\mu_{k,k}t_k(w)) = 0, (s_{k-1,k-1} - w)(\mu_{k-1,k}t_{k-1}^{(1)}(w)) + s_{k-1,k}(\mu_{k,k}t_k(w)) = 0,$$

Thus $s_{k,k}$ induces an operator $S_{k,k}$ with $\ker(S_{k,k} - w) = \bigvee \{t_k(w)\}$ and $s_{k-1,k}$ is a bundle map from $E_{t_k}(w) (:= \mathbb{C}[t_k(w)])$ to $E_{t_{k-1}}(w) (:= \mathbb{C}[t_{k-1}(w)])$.

Claim 1: For any $i \leq j \leq n-1$, $s_{i,j}$ is a bundle map from E_{t_j} to $\mathcal{J}_{j-i-1}E_{t_i}$ and there exists $m_{i,j} \in \mathbb{C}$ such that $S_{i,j}(t_j(w)) = m_{i,j}t_i^{(j-i-1)}(w)$, $w \in \mathbb{D}$.

Since $(s_{0,0} - w)\gamma_1(w) = (s_{0,0} - w)(\mu_{0,1}t_0^{(1)}(w)) + s_{0,1}(\mu_{1,1}t_1(w)) = 0$, we have

$$s_{0,1}(t_1(w)) = m_{0,1}t_0(w),$$

where $m_{0,1} = -\frac{\mu_{0,1}}{\mu_{1,1}}$. Similarly, $(s_{0,0} - w)(\mu_{0,2}t_0^{(2)}(w)) + s_{0,1}(\mu_{1,2}t_1^{(1)}(w)) + s_{0,2}(\mu_{2,2}t_2(w)) = 0$, and we have

$$s_{0,2}(t_2(w)) = -\frac{2\mu_{0,2} + \mu_{1,2}m_{0,1}}{\mu_{2,2}}t_0^{(1)}(w) = m_{0,2}t_0^{(1)}(w).$$

Now assume that for any fixed k and some $k < j \leq n-1$, there exists $m_{k,i} \in \mathbb{C}$ such that

$$s_{k,i}(t_i(w)) = m_{k,i}t_k^{(i-k-1)}(w), i < j.$$

Then from equation (2.2), we have

$$(s_{k,k} - w)(\mu_{k,j}t_k^{(j-k)}(w)) + s_{k,k+1}(\mu_{k+1,j}t_{k+1}^{(j-k-1)}(w)) + \cdots + s_{k,j}(\mu_{j,j}t_j(w)) = 0$$

and from the induction hypothesis, we may rewrite this as

$$\mu_{k,j}(j-k)t_k^{(j-k-1)}(w) + \mu_{k+1,j}m_{k,k+1}t_k^{(j-k-1)}(w) + \cdots + \mu_{j,j}s_{k,j}(t_j(w)) = 0.$$

Thus

$$s_{k,j}(t_j(w)) = m_{k,j}t_k^{(j-k-1)}(w),$$

or, equivalently

$$(2.3) \quad m_{k,j} = -\frac{\mu_{k,j}(j-k) + \sum_{l=1}^{j-k-1} \mu_{k+l,j}m_{k,k+l}}{\mu_{j,j}}$$

completing the proof of our claim.

Now that we have found constants $m_{i,j} \in \mathbb{C}$ such that

$$S_{i,j}(t_j) = m_{i,j}t_i^{(j-i-1)}, i < j = 0, 1, \dots, n-1$$

and since $(S_{i,i} - w)(t_i(w)) = 0$, $w \in \Omega$, it follows that

$$\begin{aligned} S_{i,i}S_{i,i+1}(t_{i+1}(w)) &= S_{i,i}(m_{i,i+1}t_i(w)) \\ &= m_{i,i+1}S_{i,i}(t_i(w)) \\ &= m_{i,i+1}t_i(w) \\ &= S_{i,i+1}S_{i+1,i+1}(t_{i+1}(w)). \end{aligned}$$

We have $\mathcal{H}_i = \text{Span}_{w \in \Omega}\{t_i(w)\}$, $i = 0, 1, \dots, n-1$, therefore

$$S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}, i = 0, 1, \dots, n-2.$$

This completes the proof of the second statement of the lemma.

Claim 2: For any operator T in $B_n(\Omega)$ with atomic decomposition exactly as in the second statement of the lemma, there exists $\mu_{i,j}$ satisfying the conditions set in the equations (2.1), that is, there exists a holomorphic frame for E_T , which is a linear combination of the non-vanishing holomorphic sections of E_{t_i} and a certain number of jets.

Indeed, the proof of Claim 1 already verifies Claim 2 for $n \leq 2$. To prove Claim 2 by induction, let us assume that it is valid for $k \leq n-2$. Note that the operator $(S_{i,j})_{i,j \leq n-2}$ is in $B_{n-1}(\Omega)$. By the induction hypothesis, we can find $m_{i,j}$, $i, j \leq n-2$ verifying Claim 2 for any operator $(S_{i,j})_{i,j \leq n-2}$. If we consider the operator

$$\begin{pmatrix} T_{n-2} & S_{n-2,n-1} \\ 0 & T_{n-1} \end{pmatrix},$$

then we have that $S_{n-2,n-1}(t_{n-1}) = m_{n-2,n-1}t_{n-2}$. Now, setting $\mu_{n-2,n-1} = -m_{n-2,n-1}$, we can define all the coefficients $\mu_{n-k,n-1}$, $2 \leq k \leq n$ recursively. In fact, if we consider

$$\begin{pmatrix} T_{n-k} & S_{n-k,n-k+1} & S_{n-k,n-k+2} & \cdots & S_{n-k,n-1} \\ & T_{n-k+1} & S_{n-k+1,n-k+2} & \cdots & S_{n-k+1,n-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & 0 & & & T_{n-2} & S_{n-2,n-1} \\ & & & & & T_{n-1} \end{pmatrix},$$

where $2 \leq k \leq n$, and set

$$\mu_{n-k,n-1} = -\frac{\sum_{i=1}^{k-2} m_{n-k,n-k+i} \mu_{n-k+i,n-1} + m_{n-k,n-1}}{k-1},$$

then $\mu_{n-k,n-1}$ is defined involving only the coefficients $\mu_{n-k+i,n-1}$ which exist by the induction hypothesis. Thus, coefficients $\mu_{i,j}$ depends only on the $m_{i,j}$, $i, j \leq n-1$. By a direct computation, $\gamma_k = \mu_{0,k}t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \cdots + \mu_{k,k}t_k$, $0 \leq k < n-1$ together defines a frame for E_T . This completes the proof of Claim 2 and the third statement of the lemma. \square

2.3. Boundedness. Having shown that a holomorphic quasi-homogeneous curve t defines a linear transformation on a dense subset of \mathcal{H}_t , we determine when it extends to a bounded linear operator on all of \mathcal{H}_t . We make the following conventions here which will be in force throughout this paper.

2.3.1. Conventions. The positive definite kernel $K^{(\lambda)}(z, w)$ is the function $(1 - \bar{w}z)^{-\lambda}$ defined on $\mathbb{D} \times \mathbb{D}$ and is the reproducing kernel for the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The coefficient $a_n(\lambda)$ of $\bar{w}^n z^n$ in the power series expansion for $K^{(\lambda)}$ (in powers of $z\bar{w}$) is of the form

$$\begin{aligned} a_n(\lambda) &= \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{n!} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)n\Gamma(n)} \\ &\sim n^{\lambda-1}, \quad (a_0(\lambda) = 1), \end{aligned}$$

where we have used the well-known formula due to Stirling, namely, $\frac{\Gamma(\lambda+n)}{\Gamma(n)} \sim n^\lambda$.

The set of vectors $e_n^{(\lambda)} := \sqrt{a_n(\lambda)} z^n$, $n \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The action of the multiplication operator on $\mathbb{A}^{(\lambda)}(\mathbb{D})$ is easily determined:

$$\begin{aligned} M(e_n^{(\lambda)}) &= z(\sqrt{a_n(\lambda)} z^n) \\ &= \frac{\sqrt{a_n(\lambda)}}{\sqrt{a_{n+1}(\lambda)}} \sqrt{a_{n+1}(\lambda)} z^{n+1} \\ &= \frac{\sqrt{a_n(\lambda)}}{\sqrt{a_{n+1}(\lambda)}} e_{n+1}^{(\lambda)} \\ &\sim \left(\frac{n}{n+1}\right)^{\frac{\lambda-1}{2}} e_{n+1}^{(\lambda)}. \end{aligned}$$

Often, one sets $w_n^{(\lambda)} := \frac{\sqrt{a_n(\lambda)}}{\sqrt{a_{n+1}(\lambda)}}$ and says that M is a weighted shift with weights $w_n^{(\lambda)}$ since $M(e_n^{(\lambda)}) = w_n^{(\lambda)} e_{n+1}^{(\lambda)}$. The other way round, $\prod_{i=0}^n w_i^{(\lambda)} = \sqrt{\frac{a_0(\lambda)}{a_{n+1}(\lambda)}} \sim (n+1)^{\frac{1-\lambda}{2}}$. The adjoint of this operator is then given by the formula:

$$M^*(e_n^{(\lambda)}) = w_{n-1}^{(\lambda)} e_{n-1}^{(\lambda)} \sim \left(\frac{n-1}{n}\right)^{\frac{\lambda-1}{2}} e_{n-1}^{(\lambda)}.$$

The following Lemma shows that if the valency $\Lambda(t)$ is less than 2, then every possible linear combination of the atoms and their jets need not define a bounded linear transformation. However, from the proof of this lemma, one may infer no such obstruction can occur if $\Lambda(t) \geq 2$.

Lemma 2.2. *Fix a natural number $n \geq 2$. Let t be a quasi-homogeneous holomorphic curve with atoms t_i , $i = 0, 1, \dots, n-1$. For $0 \leq i, j \leq n-1$, let $s_{i,j}(t_j(w)) = m_{i,j} t_i^{(j-i-1)}(w)$ be the bundle map from $E_{t_j}(w)$ to $\mathcal{J}_{j-i-1} E_{t_i}$ and $S_{i,j} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ be the densely defined linear transformation induced by the maps $s_{i,j}$. The linear transformation of the form*

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & T_{n-1} \end{pmatrix}$$

is densely defined on the Hilbert space $\mathbb{A}^{(\lambda_0)}(\mathbb{D}) \oplus \cdots \oplus \mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$. Suppose that $\Lambda(t) < 2$.

- (1) If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 2)$, $n \geq 2$, then T is bounded.
- (2) If $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1})$, the operator T is bounded only if we set $m_{i,j} = 0$ whenever $j-i \geq n-k-2$, $n-1 > k \geq 0$, $n \geq 4$, that is, T must be of the form

$$\begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-k-2} & 0 & \cdots & 0 & 0 \\ & S_{1,1} & \cdots & S_{1,n-k-2} & S_{1,n-k-1} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & S_{k+1,k+1} & S_{k+1,k+2} & \cdots & S_{k+1,n-1} \\ & & & & \ddots & \ddots & \vdots \\ & & & & & S_{n-2,n-2} & S_{n-2,n-1} \\ & & & & & & S_{n-1,n-1} \end{pmatrix}$$

- (3) If $\Lambda(t) \in (0, 1)$, then the densely defined linear transformation T is bounded only if we set $m_{i,j} = 0$, $i < j+1$, $i = 0, 1, \dots, n-2$, $n \geq 3$.

Proof. For $i = 0, 1, \dots, n-1$, the operators $S_{i,i}$ are homogeneous by definition. Thus the operator $S_{i,i}$, as we have said before, is realized as the adjoint of the multiplication operator on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. The reproducing kernel $K^{(\lambda_i)}(z, w)$ for this Hilbert space is of the form $\frac{1}{(1-z\bar{w})^{\lambda_i}}$. Consequently,

$$\ker(S_{i,i} - w)^* = \mathbb{C}[t_i(\bar{w})] = \mathbb{C}[K^{(\lambda_i)}(z, w)], w \in \mathbb{D}.$$

Claim : If $\lambda_j - \lambda_i > 2(j-i) - 2$, $j > i = 0, 1, 2, \dots, n-2$, then each $s_{i,j}$ induces a non-zero linear bounded operator $S_{i,j}$.

Without loss of generality, we set $s_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}$, $m_{i,j} \in \mathbb{C}$, $i, j = 0, 1, \dots, n-1$ and

$$t_i(w) = \frac{1}{(1-zw)^{\lambda_i}}, t_j(w) = \frac{1}{(1-zw)^{\lambda_j}}.$$

Then the linear transformation $S_{i,j} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ induced by $s_{i,j}$ is densely defined by the rule

$$S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}, i, j = 0, 1, \dots, n-1.$$

We have that

$$\begin{aligned} \|S_{i,j}\| &= |m_{i,j}| \max_{\ell} \left\{ \frac{\sqrt{a_{\ell}(\lambda_i)}}{\sqrt{a_{\ell-(j-i-1)}(\lambda_j)}} \ell(\ell-1) \cdots (\ell-(j-i)+2) \right\} \\ &= |m_{i,j}| \max_{\ell} \left\{ \frac{\sqrt{\prod_{l=0}^{\ell-(j-i)} w_l(\lambda_j)}}{\sqrt{\prod_{l=0}^{\ell-1} w_l(\lambda_i)}} \ell(\ell-1) \cdots (\ell-(j-i)+2) \right\} \end{aligned}$$

By a direct computation,

$$\frac{\sqrt{\prod_{l=0}^{\ell-(j-i-1)} w_l(\lambda_j)}}{\sqrt{\prod_{l=1}^{\ell-1} w_l(\lambda_i)}} \ell(\ell-1) \cdots (\ell-(j-i)+2) \sim \left(\frac{1}{\ell^{\frac{\lambda_j - \lambda_i}{2} - (j-i-1)}} \right).$$

It follows that each $S_{i,j}$ is a non-zero bounded linear operator if and only if

$$\frac{\lambda_j - \lambda_i}{2} \geq j - i - 1, \text{ that is, } \lambda_j - \lambda_i \geq 2(j - i) - 2.$$

Now recall that $T = (S_{i,j})_{n \times n}$ is of the form:

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

If $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) \geq 2(n-2).$$

By the argument given above, we obtain $S_{0,n-1}$ is non-zero and bounded. If $\Lambda(t) < 1 + \frac{n-3}{n-1}$, then we might deduce that $m_{0,n-1} = 0$ or $\mu_{0,n-1} = 0$, i.e. $S_{0,n} = 0$. Thus the proof of the first statement is complete.

For the general case, if $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1})$, $k \geq 0$, then we have

$$(n-k-1)\Lambda(t) < 2(n-k-1) - 2.$$

And if $j-i \geq n-k-1$, then we obtain

$$\begin{aligned} \lambda_j - \lambda_i &= (j-i)\Lambda(t) \\ &\leq (j-i) \frac{2(n-k-1)-2}{n-k-1} \\ &\leq (j-i) \frac{2(j-i)-2}{j-i} \\ &= 2(j-i) - 2. \end{aligned}$$

By the argument above, we have $S_{i,j} = 0, j-i \geq n-k-1$. And S has the following matrix form:

$$(2.4) \quad T = \begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-k-2} & 0 & \cdots & 0 & 0 \\ & S_{1,1} & \cdots & S_{1,n-k-2} & S_{1,n-k-1} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & & S_{k+1,k+1} & S_{k+1,k+2} & \cdots & S_{k+1,n-1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & S_{n-2,n-2} & S_{n-2,n-1} \\ & & & & & & & S_{n-1,n-1} \end{pmatrix}$$

This completes the proof of the second statement.

In particular, if $0 \leq \Lambda(t) < 1$ and $j - i \geq 2$, then we have

$$\begin{aligned}\lambda_j - \lambda_i &= (j - i)\Lambda(t) \\ &< (j - i) \\ &\leq 2(j - i) - 2,\end{aligned}$$

which implies

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & \cdots & 0 \\ 0 & S_{1,1} & S_{1,2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}, \quad \Lambda(t) \in [0, 1).$$

This completes the proof of the third statement. \square

2.4. Rigidity. Let T and \tilde{T} be two quasi-homogeneous operators acting on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$ with atomic decompositions $(S_{i,j})$ and $(\tilde{S}_{i,j})$, respectively. For a bounded linear operators X on the Hilbert space \mathcal{H} , the following statements are equivalent by definition.

- (1) $X(S_{i,j})_{n \times n} = (\tilde{S}_{i,j})_{n \times n} X$;
- (2) $X t(w) \subseteq \tilde{t}(w)$, $w \in \Omega$.

It will be convenient to separately state as a lemma what we have already proved in the third statement of Lemma 2.1.

Lemma 2.3. *Suppose T is a quasi-homogeneous operator and $(S_{i,j})_{n \times n}$ is its atomic decomposition. Then we have*

$$S_{i,i} S_{i,i+1} = S_{i,i+1} S_{i+1,i+1}, \quad i = 0, 1, \dots, n-2.$$

Recall that if A and B are two operators in $\mathcal{L}(\mathcal{H})$, then the Rosenblum operator $\tau_{A,B}$ is defined to be the operator $\tau_{A,B}(X) = AX - XB$, $X \in \mathcal{L}(\mathcal{H})$. If $A = B$, then we set $\sigma_A := \tau_{A,B}$. An operator is said to be quasi-nilpotent if $\lim_{n \rightarrow \infty} \|T^n\| = 0$. We will make repeated use of the following lemma, which appears as problem 232 in [11].

Lemma 2.4. *Let $P, T_0 \in \mathcal{L}(\mathcal{H})$ and σ_{T_0} denote the Rosenblum's operator. If $P \in \text{ran } \sigma_{T_0}$ and P commutes with T_0 , then P is quasi-nilpotent.*

Lemma 2.5. *Let P be a bounded linear operator on a Hilbert space \mathcal{H} and $t, \tilde{t} : \Omega \rightarrow \text{Gr}(1, \mathcal{H})$ be holomorphic curves. If $P t(w) = \tilde{t}(w)$, $w \in \Omega$, then either P is zero or range of P is dense.*

Proof. Suppose that P is non-zero. Let t, \tilde{t} be two holomorphic curves of rank 1. Suppose that $t(w) = \mathbb{C}[\gamma(w)]$ and $\tilde{t}(w) = \mathbb{C}[\tilde{\gamma}(w)]$, $w \in \Omega$. Now, $\gamma, \tilde{\gamma}$ are holomorphic functions taking values in the Hilbert space \mathcal{H} by definition. We claim that for any sequence $\{w_n\} \subseteq \Omega$, if $\lim_{n \rightarrow \infty} w_n = w_0 \in \text{int}(\Omega)$, then we have

$$\bigvee \{\gamma(w_n) : n \in \mathbb{N}\} = \mathcal{H}.$$

In fact, any $x \in \mathcal{H}$ which is orthogonal to $\bigvee \{\gamma(w_n) : n \in \mathbb{N}\}$ must be zero. This follows since $\langle \gamma(w), x \rangle$ is holomorphic. Pick a sequence $\{w_n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} w_n = w_0$. Then $S := \{w_i | P(\gamma(w_i)) = 0\}$ must be finite set. Otherwise, we have $\{\gamma(w_n) | w_n \in S\} = \mathcal{H}$ making $P = 0$, which is a contradiction. So there exists a subsequence $\{w_{n_k}\}_{k=1}^\infty$ and $a_{n_k} \neq 0$ such that $P(w_{n_k}) = a_{n_k} \tilde{\gamma}(w_{n_k})$. Then it follows that

$$\bigvee_{k \in \mathbb{N}} \{P(\gamma(w_{n_k}))\} = \bigvee_{k \in \mathbb{N}} \{a_{n_k} \tilde{\gamma}(w_{n_k})\} = \mathcal{H}.$$

Thus P has dense range. \square

Let t and \tilde{t} be two holomorphic curves in the Grassmannian of rank n in some Hilbert space \mathcal{H} . If there exists injective operators X and Y such that $Xt = \tilde{t}$ and $Y\tilde{t} = t$, then t and \tilde{t} are said to be quasi similar. The following lemma shows that if two quasi-homogeneous holomorphic curves t and

\tilde{t} are quasi similar via the operators X and Y , then these operators must be upper triangular with respect to the atomic decomposition of t and \tilde{t} .

Lemma 2.6. *Let t and \tilde{t} be two quasi-homogeneous holomorphic curves with atomic decomposition $\{t_i : i = 0, 1, \dots, n-1\}$ and $\{\tilde{t}_i : i = 0, 1, \dots, n-1\}$, respectively. If they are quasi-similar via the intertwining operators X and Y , that is, $Xt = \tilde{t}$ and $Y\tilde{t} = t$, then for $i \leq n-1$, we have*

$$X\left(\bigvee\{t_0(w), t_1(w), \dots, t_i(w) : w \in \Omega\}\right) \subseteq \bigvee\{\tilde{t}_0(w), \tilde{t}_1(w), \dots, \tilde{t}_i(w) : w \in \Omega\},$$

$$Y\left(\bigvee\{\tilde{t}_0(w), \tilde{t}_1(w), \dots, \tilde{t}_i(w) : w \in \Omega\}\right) \subseteq \bigvee\{t_0(w), t_1(w), \dots, t_i(w) : w \in \Omega\}.$$

Proof. We give the proof for the case of $n = 2$. By Lemma 2.2, we may assume that the holomorphic curves t and \tilde{t} define quasi-homogeneous bounded operators T and \tilde{T} , respectively. Now, Lemma 2.1 gives an atomic decomposition, say $\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{11} \end{pmatrix}$ and $\begin{pmatrix} \tilde{S}_{0,0} & \tilde{S}_{0,1} \\ 0 & \tilde{S}_{1,1} \end{pmatrix}$ for these operators. Assume that X and Y are of the form

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}, Y = \begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{pmatrix},$$

with respect to the atomic decomposition of T and \tilde{T} , respectively. We have to only show that X and Y are upper-triangular. By hypothesis, we have that

$$\begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{11} \end{pmatrix} = \begin{pmatrix} \tilde{S}_{0,0} & \tilde{S}_{0,1} \\ 0 & \tilde{S}_{1,1} \end{pmatrix} \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix},$$

and

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{11} \end{pmatrix} \begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{pmatrix} = \begin{pmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{pmatrix} \begin{pmatrix} \tilde{S}_{0,0} & \tilde{S}_{0,1} \\ 0 & \tilde{S}_{1,1} \end{pmatrix}.$$

Consequently,

$$X_{2,1}S_{0,0} = \tilde{S}_{1,1}X_{2,1},$$

$$X_{2,1}S_{0,1} + X_{2,2}S_{1,1} = \tilde{S}_{1,1}X_{2,2},$$

and

$$S_{1,1}Y_{2,1} = Y_{2,1}\tilde{S}_{0,0}.$$

From the intertwining relationships guaranteed by Lemma 2.1, $S_{0,0}S_{0,1} = S_{0,1}S_{1,1}$, and $\tilde{S}_{0,0}\tilde{S}_{0,1} = \tilde{S}_{0,1}\tilde{S}_{1,1}$, it follows that

$$X_{2,1}S_{0,1}Y_{2,1} + X_{2,2}S_{1,1}Y_{2,1} = \tilde{S}_{1,1}X_{2,2}Y_{2,1},$$

$$X_{2,1}S_{0,1}Y_{2,1} + X_{2,2}Y_{2,1}\tilde{S}_{0,0} = \tilde{S}_{1,1}X_{2,2}Y_{2,1}.$$

Furthermore,

$$X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} + X_{2,2}Y_{2,1}\tilde{S}_{0,0}\tilde{S}_{0,1} = \tilde{S}_{1,1}X_{2,2}Y_{2,1}\tilde{S}_{0,1},$$

$$X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} + X_{2,2}Y_{2,1}\tilde{S}_{0,1}\tilde{S}_{1,1} = \tilde{S}_{1,1}X_{2,2}Y_{2,1}\tilde{S}_{0,1},$$

$$X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} = \tilde{S}_{1,1}X_{2,2}Y_{2,1}\tilde{S}_{0,1} - X_{2,2}Y_{2,1}\tilde{S}_{0,1}\tilde{S}_{1,1}.$$

Thus

$$X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} \in \text{ran } \sigma_{\tilde{S}_{1,1}}.$$

We also have

$$\begin{aligned} X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1}\tilde{S}_{1,1} &= X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,0}\tilde{S}_{0,1} \\ &= X_{2,1}S_{0,1}S_{1,1}Y_{2,1}\tilde{S}_{0,1} \\ &= X_{2,1}S_{0,0}S_{0,1}Y_{2,1}\tilde{S}_{0,1} \\ &= \tilde{S}_{1,1}X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} \end{aligned}$$

showing that

$$X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} \in \mathcal{A}'(\tilde{S}_{1,1}).$$

We conclude, using Lemma 2.4, that the operator $X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1}$ is quasi-nilpotent. Note that $\tilde{S}_{1,1}$ is a homogeneous operator, which therefore must be unitarily equivalent to the adjoint of the multiplication operator on the weighted Bergman space with reproducing kernel

$$K^{(\lambda_1)}(z, w) = \frac{1}{(1 - z\bar{w})^{\lambda_1}}, \quad \lambda_1 > 0.$$

Since the operator $X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1}$ commutes with $\tilde{S}_{1,1}$, applying Lemma 2.5, we conclude that $X_{2,1}S_{0,1}Y_{2,1}\tilde{S}_{0,1} = 0$. Note that each $S_{0,1}, \tilde{S}_{0,1}, X_{0,1}$ and $Y_{0,1}$ is an intertwining operator between two holomorphic curves of rank one. Therefore, Lemma 2.5 shows that if any one of these operators is non-zero, then it must have dense range. Since $S_{0,1}$ and $\tilde{S}_{0,1}$ are both non-zero operators, we have that $X_{2,1} = 0$, or $Y_{2,1} = 0$. Without loss of generality, we suppose that $X_{2,1} = 0$. Given that $XS = \tilde{S}X$ and $SY = Y\tilde{S}$, we have

$$SYX = Y\tilde{S}X, \quad \text{and} \quad XSY = \tilde{S}XY.$$

Then we also have

$$SYX = YXS, \quad \text{and} \quad XY\tilde{S} = \tilde{S}XY.$$

So we conclude that both XY and YX are upper triangular.

Since X is upper triangular, we have $X_{2,2}S_{1,1} = \tilde{S}_{1,1}X_{2,2}$. Therefore $X_{2,2}$ has dense range. Since XY and YX are both upper triangular, we see that $X_{2,2}Y_{2,1} = 0$. Since $X_{2,2}$ has dense range, it follows that $Y_{2,1} = 0$.

The proof is now completed by induction on the rank n in pretty much the same way as in the proof of Proposition 3.2 given in [16]. \square

Repeating the proof of Lemma 2.6, we can obtain the following lemma.

Lemma 2.7. *Let E_t be a quasi-homogeneous bundle. If $X \in \mathcal{A}'(E_t)$, then X is upper-triangular.*

2.5. The Second fundamental form. In [9, page. 2244], an explicit formula for the second fundamental form of a holomorphic Hermitian line bundle in its first order jet bundle of rank 2 was given. The second fundamental form, in a slightly different guise, was shown to be a unitary invariant for the class of operators $\tilde{\mathcal{F}}B_n(\Omega)$ in [16]. We give the computation of the second fundamental form here, yet again, keeping track of certain constants which appear in the description of the quasi-homogeneous operators. We compute the second fundamental form of the inclusion E_0 in E , where $\{\gamma_0, \gamma_1\}$ is a frame for E with atoms t_0 and t_1 . The line bundle defined by the atom t_0 is E_0 . By necessity, we have

$$\gamma_0 = t_0 \quad \gamma_1 = \mu_{01}t'_0 + t_1$$

with $t_0 \perp t_1$. As in [9, 16], setting $h = \langle \gamma_0, \gamma_0 \rangle$, the second fundamental form θ_{01} is seen to be of the form

$$\theta_{01} = -h^{1/2} \frac{\bar{\partial}(h^{-1}\langle \gamma_1, \gamma_0 \rangle)}{(\|\gamma_1\|^2 - \frac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2})^{1/2}}.$$

It is important, for what follows, to express θ_{01} in terms of the atoms t_0 and t_1 giving the formula

$$(2.5) \quad \theta_{01} = \frac{\mu_{01}\mathcal{K}_0}{(\frac{\|t_1\|^2}{\|t_0\|^2} - |\mu_{01}|^2\mathcal{K}_0)^{1/2}},$$

where \mathcal{K}_0 is the curvature of the line bundle E_{t_0} given by the formula $-\bar{\partial}\partial \log \|t_0\|^2$. The following lemma shows the key role of the second fundamental form in determining the unitary equivalence class of a quasi-homogeneous holomorphic curve.

Lemma 2.8. *Suppose that t and \tilde{t} are quasi-holomorphic curves with the same atoms t_0, t_1 . Then the following statements are equivalent.*

- (1) *The two curves t and \tilde{t} are unitarily equivalent;*
- (2) *The second fundamental forms θ_{01} and $\tilde{\theta}_{01}$ are equal;*

(3) The two constants $\mu_{0,1}$ and $\tilde{\mu}_{0,1}$ are equal.

Proof. The equivalence of the first two statements was proved in [16, Corollary 2.8]. The equality of θ_{01} and $\tilde{\theta}_{01}$ is clearly equivalent to

$$\tilde{\mu}_{01} \left(\frac{\|t_1\|^2}{\|t_0\|^2} + |\mu_{01}|^2 \bar{\partial} \partial \log \|t_0\|^2 \right)^{1/2} = \mu_{01} \left(\frac{\|t_1\|^2}{\|t_0\|^2} + |\tilde{\mu}_{01}|^2 \bar{\partial} \partial \log \|t_0\|^2 \right)^{1/2}.$$

From this equality, we infer that $\arg(\mu_{01}) = \arg(\tilde{\mu}_{01})$.

Now, squaring both sides and then taking the difference, we have

$$\frac{\|t_1\|^2}{\|t_0\|^2} (\tilde{\mu}_{01}^2 - \mu_{01}^2) - \tilde{\mu}_{01}^2 \mu_{01}^2 (\bar{\partial} \partial \log \|t_0\|^2) (\tilde{\mu}_{01}^2 - \mu_{01}^2) = 0.$$

Given that we have assumed, without loss of generality, $\|t_0\|^2 = (1 - |w|^2)^{-\lambda_0}$ and $\|t_0\|^2 = (1 - |w|^2)^{-\lambda_1}$, we find that

$$\bar{\partial} \partial \log \|t_0\|^2 = \lambda_0 (1 - |w|^2)^{-2},$$

which can be equal to $\frac{\|t_1\|^2}{\|t_0\|^2}$ if and only if $\lambda_1 - \lambda_0 = 2$. Thus except when $\Lambda(t) = 2$, we must have $\mu_{01}^2 - \tilde{\mu}_{01}^2 = 0$. Clearly, $\tilde{\mu}_{01} = -\mu_{01}$ is not an admissible solution. So, we must have $\tilde{\mu}_{01} = \mu_{01}$. In case $\lambda_1 - \lambda_0 = 2$, if we assume $\tilde{\mu}_{01} \neq \mu_{01}$, then we must have

$$\left(\frac{1 + \lambda_0 |\tilde{\mu}_{01}|^2}{1 + \lambda_0 |\mu_{01}|^2} \right)^{\frac{1}{2}} = \frac{|\tilde{\mu}_{01}|}{|\mu_{01}|},$$

from which it follows that $|\tilde{\mu}_{01}| = |\mu_{01}|$. The arguments of these complex numbers being equal, they must be actually equal. \square

When we consider the inclusion of the line bundle E_{t_i} in the vector bundle $E_{\{t_i, \frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\}}$ of rank 2, the situation is slightly different. This is the vector bundle which corresponds to the 2×2 operator block $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$.

Clearly, $\{t_i, -\frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\}$ is the frame for $E_{T_{i,j}}$. By the formulae above, setting temporarily $\gamma_0 = t_i, \gamma_1 = -\frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j$, we have that

- (1) $h_i = \|\gamma_0\|^2 = \|t_i\|^2, h_j = \|t_j\|^2;$
- (2) $\|\gamma_1\|^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i} \bar{\partial}^{j-i} \|t_i\|^2 + \|t_j\|^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i} \bar{\partial}^{j-i} h_i + h_j;$
- (3) $\langle \gamma_1, \gamma_0 \rangle = -\frac{m_{i,j}}{j-i} \partial^{j-i} \|t_i\|^2 = -\frac{m_{i,j}}{j-i} \partial^{j-i} h_i;$
- (4) $|\langle \gamma_1, \gamma_0 \rangle|^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i} h_i \bar{\partial}^{j-i} h_i.$

The second fundamental form $\theta_{i,j}$ for the inclusion $E_{t_i} \subseteq E_{\{t_i, \frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\}}$ is given by the formula

$$(2.6) \quad \theta_{i,j} = \frac{\frac{m_{i,j}}{j-i} \bar{\partial} (h_i^{-1} \partial^{j-i} h_i)}{\left(\frac{h_i}{h_i} + |\frac{m_{i,j}}{j-i}|^2 \left(\frac{h_i \partial^{j-i} \bar{\partial}^{j-i} h_i - \partial^{j-i} h_i \bar{\partial}^{j-i} h_i}{h_i^2} \right) \right)^{\frac{1}{2}}}.$$

Lemma 2.9. Let $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ and $\tilde{T}_{i,j} := \begin{pmatrix} S_{i,i} & \tilde{S}_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ with $\tilde{S}_{i,j}(t_j) = \tilde{m}_{i,j} t_i^{(j-i-1)}$. The second fundamental forms $\theta_{i,j}$ and $\tilde{\theta}_{i,j}$ of the operators $T_{i,j}$ and $\tilde{T}_{i,j}$ are equal, that is, $\theta_{i,j} = \tilde{\theta}_{i,j}$ if and only if $m_{i,j} = \tilde{m}_{i,j}$.

Proof. Without loss of generality, we will give the proof only for the case $i = 0, j = k, j \neq 1$. In this case, $\theta_{0,k} = \tilde{\theta}_{0,k}$ is equivalent to the equality:

$$\frac{\left(\frac{h_k}{h_0} + \left| \frac{m_{0,k}}{k} \right|^2 \left(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2} \right) \right)^{\frac{1}{2}}}{\left(\frac{h_k}{h_0} + \left| \frac{\tilde{m}_{0,k}}{k} \right|^2 \left(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2} \right) \right)^{\frac{1}{2}}} = \frac{m_{0,k}}{\tilde{m}_{0,k}}$$

For simplicity, let g_0 denote $(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2})$ and let m, \tilde{m} denote $\frac{m_{0,k}}{k}, \frac{\tilde{m}_{0,k}}{k}$ respectively. Then the equation given above may be rewritten as

$$\frac{(\frac{h_k}{h_0} + |m|^2 g_0)^{\frac{1}{2}}}{(\frac{h_k}{h_0} + |\tilde{m}|^2 g_0)^{\frac{1}{2}}} = \frac{m}{\tilde{m}}$$

From this equality, we infer that $\arg(m) = \arg(\tilde{m})$. Now, squaring both sides and then taking the difference, we have

$$\frac{h_k}{h_0}(\tilde{m}^2 - m^2) - \tilde{m}^2 m^2 g_0(\bar{m}^2 - \bar{\tilde{m}}^2) = 0.$$

Having assumed, without loss of generality, $h_0 = (1 - |w|^2)^{-\lambda_0}$ and $h_k = (1 - |w|^2)^{-\lambda_1}$, we find that g_0 is a polynomial of degree > 1 in $(1 - |w|^2)^{-1}$. Thus g_0 can be equal to $\frac{h_k}{h_0}$ if and only if $\lambda_1 - \lambda_0 = 2$. Therefore, except when $\Lambda(t) = 2$, we must have $m^2 - \tilde{m}^2 = 0$. Clearly, $m = -\tilde{m}$ is not an admissible solution. So, we must have $m = \tilde{m}$. Hence $m_{0,k} = \tilde{m}_{0,k}$. \square

2.6. Unitary equivalence. Recall that a positive definite kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}^{n \times n}$ is said to be normalized at $w_0 \in \Omega$, if $K(z, w_0) = I$, $z \in \Omega$. An operator T in $B_n(\Omega)$ may be realized, up to unitary equivalence, as the adjoint of a multiplication operator on a Hilbert space possessing a normalized reproducing kernel (cf. [5]). Realized in this form, the operator is determined completely modulo multiplication by a constant unitary operator acting on \mathbb{C}^n . As one might expect, finding the normalized kernel if $n > 1$ is not easy. The second statement of the theorem below is a rigidity theorem in the spirit of what was proved by Curto and Salinas for operators in $B_n(\mathbb{D})$. For quasi-homogeneous operators, the atoms are homogeneous operators in $B_1(\mathbb{D})$. These are assumed to be realized in normal form. Consequently, if T is a quasi-homogeneous operator, a set of $n - 1$ fundamental forms determine the operator T completely, that is, two of them are unitarily equivalent if and only if they are equal assuming they have the same set second fundamental forms. The first of the two statements given in the theorem below was proved for operators in $\mathcal{FB}_n(\Omega)$ (cf. [16, Proposition 3.5]). We have included it here only for the sake of completeness.

Theorem 2.10. *For any two holomorphic curves t and \tilde{t} with atoms $\{t_i : 0 \leq i \leq n - 1\}$ and $\{\tilde{t}_i : 0 \leq i \leq n - 1\}$, respectively, we have the following.*

- (1) *If t and \tilde{t} are unitarily equivalent, then for $0 \leq i \leq n - 1$,*
 - (a) $\mathcal{K}_{t_i} = \mathcal{K}_{\tilde{t}_i}$, $0 \leq i \leq n - 1$;
 - (b) $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $0 \leq i \leq n - 2$.
- (2) *Suppose that t and \tilde{t} are unitarily equivalent. Then if the second fundamental forms are the same, that is, $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $0 \leq i \leq n - 2$, then $t = \tilde{t}$.*

Proof. If necessary, conjugating by a diagonal unitary, without loss of generality, we may assume that the atoms of the operators T and \tilde{T} are the same. If there exists a unitary operator U such that $TU = U\tilde{T}$, then U must be diagonal with unitaries U_0, U_1, \dots, U_{n-1} on its diagonal. Then we have

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j, \quad i, j = 0, 1, \dots, n - 1.$$

In particular, U_i commutes with the fixed set of atoms T_i , which are irreducible, therefore there exists $\beta_i \in [0, 2\pi]$ such that

$$U_i = e^{i\beta_i} I_{\mathcal{H}_i}, \quad i = 0, 1, \dots, n - 1.$$

Then on the one hand, we have

$$U_i S_{i,i+1}(t_{i+1}) = U_i(-\mu_{i,i+1} t_i) = -\mu_{i,i+1} e^{i\beta_i} t_i$$

and on the other hand, we have

$$\tilde{S}_{i,i+1} U_{i+1}(t_{i+1}) = S_{i,i+1}(e^{i\beta_{i+1}} t_{i+1}) = -\tilde{\mu}_{i,i+1} e^{i\beta_{i+1}} t_i.$$

Consequently,

$$-\mu_{i,i+1}e^{i\beta_i} = -\tilde{\mu}_{i,i+1}e^{i\beta_{i+1}}, \quad 0 \leq i \leq n-2.$$

The assumption that the second fundamental forms are the same for the two operators T and \tilde{T} implies that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$. Therefore, we have $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $i = 0, 1, \dots, n-1$. \square

Remark 2.11. It is natural to ask which of the quasi-homogeneous operators are homogeneous. A comparison with the homogeneous operators given in [24] shows that a quasi-homogeneous operator is homogeneous if and only if

$$(2.7) \quad \mu_{i,j} = \frac{\Gamma_{i,j}(\lambda)\mu_i}{\mu_j}, \quad \Gamma_{i,j}(\lambda) = \binom{i}{j} \frac{1}{(2\lambda_j)_{i-j}}, \quad \lambda_j = \lambda - \frac{m}{2} + j,$$

for some choice of positive constants $\mu_0(=1), \mu_1, \dots, \mu_{n-1}$. Here $(\alpha)_\ell := \alpha(\alpha+1)\cdots(\alpha+\ell-1)$ is the Pochhammer symbol. Clearly, if two homogeneous operators with (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$ were unitarily equivalent, then λ must equal $\tilde{\lambda}$. Since it is easy to see that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$ if and only if $\mu_i = \tilde{\mu}_{i+1}$, we conclude two of these homogeneous operators are unitarily equivalent if and only if they are equal recovering previous results of [24].

3. CANONICAL MODEL UNDER SIMILARITY

In this section, our main focus is on the question of reducibility and strong irreducibility of a quasi-homogeneous operator. We recall that an operator T is said to be strongly irreducible if there is no idempotent in its commutant, or equivalently, there does not exist an invertible operator L for which LTL^{-1} is reducible. The (multiplicity-free) homogeneous operators in the Cowen-Douglas class of rank n are irreducible (cf. [24]). However, they were shown (cf. [23]) to be similar to the n -fold direct sum of their atoms making them strongly reducible. It is this phenomenon that we investigate here for quasi-homogeneous operators. Along the way, we determine when two quasi-homogeneous operators are similar. Our investigations show that there is dichotomy which depends on whether or not the valency $\Lambda(t)$ is less than 2 or greater or equal to 2. In what follows, we will say that a holomorphic curve $t : \mathbb{D} \rightarrow Gr(n, \mathcal{H})$ is strongly irreducible if there is no invertible operator L on the Hilbert space \mathcal{H} for which Xt splits into orthogonal direct sum of two holomorphic curves, say t_1 and t_2 , in $Gr(n_1, \mathcal{H})$ and $Gr(n_2, \mathcal{H})$, $n_1 + n_2 = n$, respectively.

Suppose $t : \mathbb{D} \rightarrow Gr(n, \mathcal{H})$ is a quasi-homogeneous holomorphic curve with atoms t_0, t_1, \dots, t_{n-1} . Then t is strongly reducible, $t \sim t_0 \oplus t_1 \cdots \oplus t_{n-1}$, if $\Lambda(t) \geq 2$ and strongly irreducible otherwise. The dichotomy involving the valency $\Lambda(t)$ is also clear from the main theorem on similarity of quasi-homogeneous holomorphic curves.

The atoms of a quasi-homogeneous operator are homogeneous operators in $B_1(\mathbb{D})$ by definition. Therefore, they are uniquely determined not only up to unitary equivalence but upto similarity as well. Now, pick any two quasi-homogeneous operators. They possess an atomic decomposition by virtue of Lemma 2.1. Any invertible operator intertwining these two quasi-homogeneous operators is necessarily upper triangular by Lemma 2.6 with respect to their respective atomic decomposition. Hence if two quasi-homogeneous operators are similar, then each of the atoms for one must be similar to the other. Consequently, to determine equivalence of quasi-homogeneous operators T under an invertible linear transformation, we may assume (as before) without loss of generality that the atoms are fixed with the weight λ_0 and the valency $\Lambda(t)$. Clearly, the valency $\Lambda(t)$ is both an unitary as well as a similarity invariant of the quasi-homogeneous curve t .

Note that if we let R be the $n \times n$ diagonal matrix with $(\prod_{\ell=0}^i \mu_{\ell, \ell+1})(\prod_{\ell=0}^i \tilde{\mu}_{\ell, \ell+1})^{-1}$ on its diagonal and set $\tilde{t} = R t R^{-1}$, then $\tilde{S}_{i,i+1}(t_{i+1}) = \tilde{\mu}_{i,i+1}$, $0 \leq i \leq n-2$. Thus up to similarity, we may assume that the constants $\mu_{i,i+1}$ and $\tilde{\mu}_{i,i+1}$ are the same. Or equivalently (see Lemma 2.8), we may assume that the choice of the second fundamental forms $\theta_{i,i+1}$, $0 \leq i \leq n-2$, does not change the similarity

class of a quasi-homogeneous holomorphic curve. Therefore the condition in the second statement of the theorem given below is not a restriction on the similarity class of the holomorphic curves t and \tilde{t} .

Theorem 3.1. *Suppose t and \tilde{t} are quasi-homogeneous holomorphic curves.*

- (1) *If $\Lambda(t) \geq 2$, then t is similar to the n -fold direct sum of the atoms $t_0 \oplus t_1 \oplus \cdots \oplus t_{n-1}$.*
- (2) *If $\Lambda(t) = \Lambda(\tilde{t}) < 2$ and $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $i = 0, 1, \dots, n-2$, then t and \tilde{t} are similar if and only if they are equal.*

In what follows, for brevity of notation, we let $T_0 := S_{0,0}$ and $T_{k+1} := S_{k+1,k+1}$ ($k, 1 \leq k \leq n-2$ is fixed but arbitrary) be the two atoms of a quasi-homogeneous operator T . As always, we assume they have been realized as the adjoint of the multiplication operators on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{k+1})}(\mathbb{D})$, respectively.

3.1. The Key Lemma. The following lemma is the key to determining when a bundle map that intertwines two quasi-homogeneous holomorphic vector bundles extends to an invertible bounded operator. It reveals the intrinsic structure of the intertwiners between two quasi-homogeneous bundles. We follow the conventions set up in Section 2.3.1.

Lemma 3.2. *Let E_t be a quasi-homogeneous vector bundle and $s_{i,j}, i, j = 0, 1, \dots, n-1$ be the induced bundle maps. There exists a bundle map $X : E_{t_{n-1}} \rightarrow \mathcal{J}_{n-1}(E_{t_0})$ with the intertwining property*

$$s_{0,0}X - XS_{n-1,n-1} = s_{0,n-1}$$

that extends to a bounded linear operator only if $\Lambda(t) \geq 2$.

Proof. Let T_0 and T_{k+1} be the operators induced by $s_{0,0}$ and $s_{k+1,k+1}$ as in Lemma 2.1. These are then necessarily the operators $M^{(\lambda_0)*}$ and $M^{(\lambda_{k+1})*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{k+1})}(\mathbb{D})$, respectively.

The kernel of the operator $(T_i - w)$, $w \in \mathbb{D}$, is spanned by the vector $t_i(w) := (1 - z\bar{w})^{-\lambda_i}$, $i = 0, k+1$. By hypothesis, for each fixed $w \in \mathbb{D}$, we have $S_{0,k+1}((1 - z\bar{w})^{-\lambda_{k+1}}) = \bar{\partial}^k(1 - z\bar{w})^{-\lambda_0}$. Differentiating both sides of this equation ℓ times and then evaluating at $w = 0$, we get $S_{0,k+1}((\lambda_{k+1})_\ell z^\ell) = (\lambda_0)_{\ell+k} z^{\ell+k}$. For $j = 0$ or $j = k-1$, the set of vectors $e_\ell^{(\lambda_j)} := \sqrt{a_\ell(\lambda_j)} z^\ell$, $\ell \geq 0$ is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$. The matrix representation for the operator $S_{0,k+1} : \mathbb{A}^{(\lambda_{k+1})}(\mathbb{D}) \rightarrow \mathbb{A}^{(\lambda_0)}(\mathbb{D})$ with respect to this orthonormal basis is obtained from the computation:

$$\begin{aligned} S_{0,k+1}(e_\ell^{(\lambda_{k+1})}) &= S_{0,k+1}(\sqrt{a_\ell(\lambda_{k+1})} z^\ell) \\ &= \sqrt{\frac{a_\ell(\lambda_{k+1})}{a_{\ell+k}(\lambda_0)}} \frac{(\lambda_0)_{\ell+k}}{(\lambda_{k+1})_\ell} \sqrt{a_{\ell+k}(\lambda_0)} z^{\ell+k} \\ &= \frac{(\ell+k)!}{\ell!} \sqrt{\frac{a_{\ell+k}(\lambda_0)}{a_\ell(\lambda_{k+1})}} e_{\ell+k}^{(\lambda_0)}. \end{aligned}$$

Thus $S_{0,k+1}$ is a forward shift of multiplicity k . We claim that if $\Lambda(t) \geq 2$, then we can find a forward shift X of multiplicity $k+1$, namely, $X(e_\ell^{(\lambda_{k+1})}) = x_\ell e_{\ell+k+1}^{(\lambda_0)}$ which has the required intertwining property. Thus evaluating the equation $S_{0,0}X - XS_{n-1,k+1} = S_{0,k+1}$ on the vectors $e_{\ell-1}^{(\lambda_{k+1})}$, $\ell \geq 0$, we obtain

$$\begin{aligned} (3.1) \quad \frac{(\ell+k)!}{\ell!} \frac{\prod_{i=0}^{\ell-1} w_i^{(\lambda_{k+1})}}{\prod_{i=0}^{\ell+k-1} w_i^{(\lambda_0)}} e_{\ell+k}^{(\lambda_0)} &= \frac{(\ell+k)!}{\ell!} \sqrt{\frac{a_{\ell+k}(\lambda_0)}{a_\ell(\lambda_{k+1})}} e_{\ell+k}^{(\lambda_0)} \\ &= S_{0,k+1}(e_\ell^{(\lambda_{k+1})}) \\ &= (S_{0,0}X - XS_{k+1,k+1})(e_\ell^{(\lambda_{k+1})}) \\ (3.2) \quad &= (x_\ell w_{\ell+k}^{(\lambda_0)} - x_{\ell-1} w_{\ell-1}^{(\lambda_{k+1})}) e_{\ell+k}^{(\lambda_0)}. \end{aligned}$$

From this we obtain x_ℓ recursively:

$$w_k^{(\lambda_0)} x_0 = k! \frac{\sqrt{a_k(\lambda_0)}}{\sqrt{a_0(\lambda_{k+1})}}$$

and for $\ell \geq 1$,

$$\begin{aligned} x_\ell &= \sqrt{\frac{a_{k+\ell}(\lambda_0)}{a_\ell(\lambda_{k+1})}} \sum_{i=1}^k (\ell)_i \sim \left((k+\ell)^{\frac{\lambda_0-1}{2}} \right) \left(\ell^{\frac{-\lambda_{k+1}+1}{2}} \right) (\ell^{k+1}) \\ &\sim \left(\ell^{\frac{\lambda_0-\lambda_{k+1}+2k+2}{2}} \right), \end{aligned}$$

where $(\ell)_k := \ell(\ell+1)\cdots(\ell+k-1) = \frac{\Gamma(\ell+k)}{\Gamma(\ell)}$ is the Pochhammer symbol as before. Here, using the Stirling approximation for the Γ function, we infer that $\sum_{i=1}^k (\ell)_i \sim \ell^{k+1}$.

If $\Lambda(t) \geq 2$, then $\lambda_1 - \lambda_0 \geq 2, \lambda_2 - \lambda_1 \geq 2, \dots, \lambda_{k+1} - \lambda_k \geq 2$. Consequently, $\lambda_{k+1} - \lambda_0 \geq 2k+2$ making the operator X bounded.

It follows that if $\Lambda(t) \geq 2$, then the shift X of multiplicity n that we have constructed is bounded and has the desired intertwining property. To show that there is no such intertwining operator if $\Lambda(t) < 2$, assume to the contrary the existence of such an operator. Then we show that there must also exist a shift of multiplicity $k+1$ with this property leading to a contradiction. For the proof, suppose

$$X(e_\ell^{(\lambda_{k+1})}) = \sum_{i=0}^{\infty} x_{i,\ell} e_i^{(\lambda_0)}, \quad X = \langle x_{i,\ell} \rangle.$$

Then

$$(S_{0,0}X - XS_{k+1,k+1})(e_\ell^{(\lambda_{k+1})}) = \sum_{i=0}^{\infty} (x_{i+1,\ell+1} w_i^{(\lambda_0)} - x_{i,\ell} w_{\ell-1}^{(\lambda_{k+1})})(e_i^{(\lambda_0)}).$$

In particular, we have

$$(x_{\ell+k+1,\ell+1} w_{\ell+k}^{(\lambda_0)} - x_{\ell+k,\ell} w_{\ell-1}^{(\lambda_{k+1})})(e_{\ell+k}^{(\lambda_0)}) = S_{0,k+1}(e_\ell^{(\lambda_{k+1})}).$$

Repeating the proof above, we will have the conclusion $x_{l+k,l} \rightarrow \infty, l \rightarrow \infty$ which proving the claim. \square

Lemma 3.3. *Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq n-1$. Let $T := \langle S_{i,j} \rangle$ be the atomic decomposition of the operator T representing t as in Lemma 2.1.*

- (1) *If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 1 + \frac{n-2}{n})$, then for any $1 \leq r < n-1$, we have*

$$S_{0,r} S_{r,r+1} \cdots S_{n-2,n-1} \in \text{ran } \sigma_{S_{0,0}, S_{n-1,n-1}}.$$

- (2) *Suppose that $\Lambda(t) \geq 2$. Then there exists a bounded linear operator $X \in \mathcal{L}(\mathcal{H}_{n-1}, \mathcal{H}_{n-2})$ such that*

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}$$

and

$$S_{n-3,n-2}X \in \text{ran } \sigma_{S_{n-3,n-3}, S_{n-1,n-1}}.$$

Proof. We only prove that $S_{0,n-2}S_{n-2,n-1}$ is in $\text{ran } \sigma_{S_{0,0}, S_{n-1,n-1}}$. Clearly, as can be seen from the proof we present below, the proof in all the other cases are exactly the same.

Let T_0, T_{n-2} and T_{n-1} be the operators induced by $s_{0,0}, s_{n-2,n-2}$ and s_{n-1} as in Lemma 2.1. These are then necessarily the operators $M^{(\lambda_0)*}, M^{(\lambda_{n-2})*}$ and $M^{(\lambda_{n-1})*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D}), \mathbb{A}^{(\lambda_{n-2})}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$, respectively.

As in the proof of Lemma 3.2, equations (3.1) and (3.2), we have that

$$\begin{aligned} S_{0,n-2}(e_\ell^{(\lambda_{n-2})}) &= S_{0,n-2}(\sqrt{a_\ell(\lambda_{n-2})} z^\ell) \\ &= \sqrt{\frac{a_\ell(\lambda_{n-2})}{a_{\ell+n-3}(\lambda_0)} \frac{(\lambda_0)_{\ell+n-3}}{(\lambda_{n-2})_\ell}} \sqrt{a_{\ell+n-3}(\lambda_0)} z^{\ell+n-3} \\ &= \frac{(\ell+n-3)!}{\ell!} \sqrt{\frac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-2})}} e_{\ell+n-3}^{(\lambda_0)} \end{aligned}$$

$$S_{n-2,n-1}(e_\ell^{(\lambda_{n-1})}) = \frac{\sqrt{a_\ell(\lambda_{n-2})}}{\sqrt{a_\ell(\lambda_{n-1})}} e_\ell^{(\lambda_{n-2})};$$

and

$$S_{0,n-2}S_{n-2,n-1}(e_\ell^{(\lambda_{n-1})}) = \frac{(\ell+n-3)!}{\ell!} \sqrt{\frac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-1})}} e_{\ell+n-3}^{(\lambda_0)}.$$

Thus $S_{0,n-2}S_{n-2,n-1}$ is a forward shift of multiplicity $n-3$. We claim that if $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then we can find a forward shift X of multiplicity $n-2$, namely, $X(e_\ell^{(\lambda_{n-1})}) = x_\ell e_{\ell+n-2}^{(\lambda_0)}$ which has the required intertwining property. Thus evaluating the equation $S_{0,0}X - XS_{n-1,n-1} = S_{0,n-1}$ on the vectors $e_\ell^{(\lambda_{n-1})}$, $\ell \geq 0$, we obtain

$$w_{n-3}^{(\lambda_0)} x_0 = (n-3)! \frac{\sqrt{a_{n-3}(\lambda_0)}}{\sqrt{a_0(\lambda_{n-1})}}$$

and for $\ell \geq 1$, we have that

$$w_{\ell+n-3}^{(\lambda_0)} x_\ell - x_{\ell-1} w_\ell^{(\lambda_{n-1})} = \frac{(\ell+n-3)!}{\ell!} \frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{n-1})}}.$$

It follows that

$$\begin{aligned} x_\ell &= \frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{n-1})}} \sum_{i=1}^{n-3} (\ell)_i \sim \left((n-3+\ell)^{\frac{\lambda_0-1}{2}} \right) \left(\ell^{\frac{-\lambda_{n-1}+1}{2}} \right) (\ell^{n-2}) \\ &\sim \left(\ell^{\frac{\lambda_0-\lambda_{n-1}+2n-4}{2}} \right). \end{aligned}$$

Note that when $\Lambda(t) > 1 + \frac{n-3}{n-1}$, we obtain

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) > (n-1)\frac{2n-4}{n-1} = 2n-4$$

making X bounded. This completes the proof of the first statement.

For the proof of the second statement, note that by virtue of Lemma 3.2, we have $S_{n-2,n-1} \in \text{Ran} \sigma_{S_{n-2,n-1}}$. So there exists a bounded operator X such that

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}.$$

Repeating the proof for the first part, we conclude

$$S_{n-3,n-2}X \in \text{ran } \sigma_{S_{n-3,n-3}, S_{n-1,n-1}}.$$

□

3.2. Strong irreducibility. We now show that a quasi-homogeneous holomorphic curve t is strongly irreducible or strongly reducible according as $\Lambda(t)$ is less than 2 or greater equal to 2. We recall that homogeneous operators (in this case, $\Lambda(t) = 2$) were shown to be irreducible but strongly reducible in [23]

Lemma 3.4. *Fix a quasi-homogeneous holomorphic curve t with atoms t_i and let $T = \langle\langle S_{i,j} \rangle\rangle$ be its atomic decomposition.*

- (1) *If $\Lambda(t) \geq 2$, then T is strongly reducible, indeed T is similar to the direct sum of its atoms, namely, $\bigoplus_{i=0}^{n-1} T_i$ and*
- (2) *if $\Lambda(t) < 2$, then T is strongly irreducible.*

Proof. If $\Lambda(t) \geq 2$, then we claim that the operator T is similar to $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$.

When $n = 2$, Let $T = \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}$. By Lemma 3.2, there exists $X_{0,1}$ such that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = S_{0,1}.$$

Set $Y_{0,1} = \begin{pmatrix} I & X_{0,1} \\ 0 & I \end{pmatrix}$, then we have that

$$\begin{aligned} Y_{0,1}TY_{0,1}^{-1} &= \begin{pmatrix} S_{0,0} & S_{0,1} + X_{0,1}S_{1,1} \\ 0 & S_{1,1} \end{pmatrix} \begin{pmatrix} I - X_{0,1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} S_{0,0} & S_{0,1} - S_{0,0}X_{0,1} + X_{0,1}S_{1,1} \\ 0 & S_{1,1} \end{pmatrix} \\ &= \begin{pmatrix} S_{0,0} & 0 \\ 0 & S_{1,1} \end{pmatrix} \end{aligned}$$

Notice that $Y_{0,1}$ is invertible, we have that $T \sim S_{0,0} \oplus S_{1,1}$.

In this case, using Lemma 3.2, we find an invertible bounded linear operator $X_{0,n-1}$ such that

$$S_{0,0}X_{0,n-1} - X_{0,n-1}S_{n-1,n-1} = S_{0,n-1}.$$

For any $i < j$, applying Lemma 3.2 to the operators

$$\begin{pmatrix} S_{i,i} & S_{i,i+1} & S_{i,i+2} & \cdots & S_{i,j} \\ 0 & S_{i+1,i+1} & S_{i+1,i+2} & \cdots & S_{i+1,j} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{j-1,j-1} & S_{j-1,j} \\ 0 & 0 & \cdots & 0 & S_{j,j} \end{pmatrix},$$

we find an invertible bounded linear operator $X_{i,j}$ such that $S_{i,i}X_{i,j} - X_{i,j}S_{j,j} = S_{i,j}$. Set $Y_{n-2,n-1} := \left(\begin{array}{c|c} I^{(n-2)} & 0 \\ \hline 0 & I \end{array} \begin{array}{c} X_{n-2,n-1} \\ I \end{array} \right)$ and note that $Y_{n-2,n-1}^{-1} = \left(\begin{array}{c|c} I^{(n-2)} & 0 \\ \hline 0 & I \end{array} \begin{array}{c} -X_{n-2,n-1} \\ I \end{array} \right)$. Now, we have

$$\begin{aligned} &\left(\begin{array}{c|c} I^{(n-2)} & 0 \\ \hline 0 & I \end{array} \begin{array}{c} X_{n-2,n-1} \\ I \end{array} \right) \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} \left(\begin{array}{c|c} I^{(n-2)} & 0 \\ \hline 0 & I \end{array} \begin{array}{c} -X_{n-2,n-1} \\ I \end{array} \right) \\ &= \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} - S_{0,n-2}X_{n-2,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1} - S_{n-3,n-2}X_{n-2,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}. \end{aligned}$$

By Lemma 3.3, we have

$$S_{n-3,n-2}X_{n-2,n-1} \in \text{ran } \sigma_{S_{n-1,n-1}, S_{n-3,n-3}}.$$

Therefore, there exists an invertible bounded linear operator \tilde{X} such that

$$S_{n-3,n-3}\tilde{X} - \tilde{X}S_{n-1,n-1} = S_{n-3,n-1} - S_{n-3,n-2}X_{n-2,n-1}.$$

Let $X_{n-3,n-1} := \tilde{X}$ and $Y_{n-3,n-1} = \left(\begin{array}{c|ccc} I^{(n-3)} & 0 & & \\ \hline 0 & I & 0 & X_{n-3,n-2} \\ & 0 & I & 0 \\ & 0 & 0 & I \end{array} \right)$. Now, we have

$$\begin{aligned} Y_{n-3,n-1} & \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} - S_{0,n-2}X_{n-2,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1} - S_{n-3,n-2}X_{n-2,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} Y_{n-3,n-1}^{-1} \\ &= \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} - S_{0,n-2}X_{n-2,n-1} - S_{0,n-3}X_{n-3,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & \cdots & 0 \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}. \end{aligned}$$

Continuing in this manner, we clearly have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} \sim \begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-2} & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & 0 \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

This completes the proof of the induction step. We have therefore proved the first statement.

To prove the second statement, assuming that $\Lambda(t) < 2$, we must show that E_t is strongly irreducible. First, we prove that E_t is irreducible. By Lemma 2.6, any projection $P = (P_{i,j})_{n \times n}$ in $\mathcal{A}'(E_t)$ is diagonal. Thus

$$P_{i,i}^2 = P_{i,i} \in \mathcal{A}'(E_{t_i}).$$

It follows that for any $0 \leq i \leq n-1$, $P_{i,i} = 0$ or $P_{i,i} = I$. Since $PS = SP$, we have

$$P_{i,i}S_{i,i+1} = S_{i,i+1}P_{i+1,i+1}.$$

Therefore

$$P_{i,i} = P_{j,j}, i, j = 0, 1, \dots, n-1.$$

Consequently, $P = 0$ or $P = I$ and E_t is irreducible.

We first prove that E_t is also strongly irreducible for $n = 2$. By Lemma 3.2, we have

$$S_{0,1} \notin \text{ran } \sigma_{S_{0,0}, S_{1,1}}.$$

Let $P \in \mathcal{A}'(E_t)$ be an idempotent. By Lemma 2.3, P has the following form

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} \\ 0 & P_{1,1} \end{pmatrix}.$$

Since $PS = SP$, we have

$$P_{0,0}S_{0,0} = S_{0,0}P_{0,0}, P_{1,1}S_{1,1} = S_{1,1}P_{1,1}$$

and

$$P_{00}S_{0,1} - S_{0,1}P_{11} = S_{0,0}P_{0,1} - P_{0,1}S_{1,1}.$$

Since $P_{i,i} \in \{S_{i,i}\}'$, for $0 \leq i \leq 1$, so $P_{i,i}$ can be either I or 0 . If either $P_{1,1} = I$, $P_{2,2} = 0$ or $P_{0,0} = 0$, $P_{1,1} = I$, then $S_{0,1} \in \text{Ran } \sigma_{S_{0,0}, S_{1,1}}$ which is a contradiction to our conclusion that $S \notin \text{ran } \sigma_{S_{0,0}, S_{1,1}}$. Thus the form of P will be

$$\begin{pmatrix} I & P_{0,1} \\ 0 & I \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & P_{0,1} \\ 0 & 0 \end{pmatrix}.$$

Since P is an idempotent operator, so we have $P_{0,1} = 0$. Hence E_t is strongly irreducible.

To complete the proof of the second statement by induction, suppose that it is valid for any $n \leq k-1$. For $n = k$, let $P \in \mathcal{A}'(E_t)$ be an idempotent operator. By Lemma 2.6, P has the following form:

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots & P_{0,k} \\ 0 & P_{1,1} & P_{1,2} & \cdots & P_{1,k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{k-1,k-1} & P_{k-1,k} \\ 0 & \cdots & \cdots & 0 & P_{k,k} \end{pmatrix},$$

and $P((S_{i,j}))_{k \times k} = ((S_{i,j}))_{k \times k} P$. It follows that

$$((P_{i,j}))_{i,j=0}^{k-1} ((S_{i,j}))_{i,j=0}^{k-1} = ((S_{i,j}))_{i,j=0}^{k-1} ((P_{i,j}))_{i,j=0}^{k-1}, ((P_{i,j}))_{i,j=1}^k ((S_{i,j}))_{i,j=1}^k = ((S_{i,j}))_{i,j=1}^k ((P_{i,j}))_{i,j=1}^k.$$

Both $((P_{i,j}))_{i,j=0}^{k-1}$ and $((P_{i,j}))_{i,j=1}^k$ are idempotents. Since $\Lambda(t) < 2$, we have

$$S_{r,s} \notin \text{ran } \sigma_{S_{r,r}, S_{s,s}}, \quad r, s \leq n.$$

By the induction hypothesis, we have

$$P_{i,j} = 0, i \neq j \leq k-1,$$

and

$$P_{0,0} = P_{1,1} = \cdots = P_{k,k} = 0, \text{ or } P_{0,0} = P_{1,1} = \cdots = P_{k,k} = I.$$

Thus P has the following form:

$$P = \begin{pmatrix} I & 0 & 0 & \cdots & P_{0,k} \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{pmatrix} \text{ or } P = \begin{pmatrix} 0 & 0 & 0 & \cdots & P_{0,k} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Since P is an idempotent, it follows that $P_{0,k} = 0$. □

By Lemma 2.6, an intertwining operator between two quasi-homogeneous operators with respect to any atomic decomposition must be upper triangular. Thus any operator X in the commutant of such an operator, say T , must also be upper-triangular. In particular, $X_{i,i}$ belongs to the commutant of $S_{i,i}$, $0 \leq i \leq n-1$. Since $S_{i,i}$ is a homogeneous operator in $B_1(\mathbb{D})$, it follows that the commutant of $S_{i,i}$ is isomorphic to $\mathcal{H}^\infty(\mathbb{D})$, the space of bounded analytic functions on the unit disc \mathbb{D} . Consequently, for any $\phi \in \mathcal{H}^\infty(\mathbb{D})$, the operator $\phi(S_{i,i})$ is in the commutant $\mathcal{A}'(S_{i,i})$. In the following lemma, we give a description of the commutant of T . We will construct an operator X in the commutant of T , where the diagonal elements are induced by the same holomorphic function $\phi \in \mathcal{H}^\infty(\mathbb{D})$, that is, $\phi(S_{i,i}) = X_{i,i}$.

Lemma 3.5. *Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq 1$. Let $T = ((S_{i,j}))$ be its atomic decomposition. Suppose that $X = ((X_{i,j}))$ is in $\mathcal{A}'(T)$. Then there exists $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \phi(S_{i,i}), i = 0, 1$ and we also have that*

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0.$$

In particular, $X_{0,1}$ can be chosen as zero.

Proof. Set $X = ((X_{i,j})) \in \mathcal{A}'(T)$, we have the following equation

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}.$$

By Lemma 2.7, we have $X_{1,0} = 0$. Then

$$S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1},$$

and

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}.$$

Note that there exist holomorphic functions $\phi_{0,0}$ and $\phi_{1,1}$ such that

$$X_{0,0}(t_0) = \phi_{0,0}(t_0), X_{1,1}(t_1) = \phi_{1,1}(t_1),$$

and by the definition of $S_{0,1}$, there exist constant function $\phi_{0,1}$ such that

$$S_{0,1}(t_1) = \phi_{0,1}t_0.$$

Then

$$X_{0,0}S_{0,1}(t_1) - S_{0,1}X_{1,1}(t_1) = (\phi_{0,0}\phi_{0,1} - \phi_{1,1}\phi_{0,1})t_0.$$

and $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ also intertwines $S_{0,0}$ and $S_{1,1}$. Taking $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ the place of $S_{0,1}$ and using the proof of Lemma 3.2, we might deduce that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0, \phi_{0,0} = \phi_{1,1}.$$

Thus, we can choose $X_{0,1} = 0$ and there exists $\phi = \phi_{0,0} = \phi_{1,1} \in \mathcal{H}^\infty(\mathbb{D})$ such that $X = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix}$ where $X_{i,i} = \phi(S_{i,i})$ satisfies that

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}.$$

□

Lemma 3.6. *Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq n-1$. Let $T = \langle\langle S_{i,j} \rangle\rangle$ be its atomic decomposition. Let $\phi \in \mathcal{H}^\infty(\mathbb{D})$ be a holomorphic function. If $\Lambda(t) < 2$, then there exists a bounded linear operator $X \in \mathcal{A}'(T)$ such that $X_{i,i} = \phi(S_{i,i}), i = 0, 1, \dots, n-1$.*

Proof. Firstly, by Lemma 3.5, the lemma is true for the case of $n = 2$.

For $n = 3$, let $X = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \in \mathcal{A}'(E_t)$. Then we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}$$

and it follows that

- (1) $S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1}$, that is, $S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$;
- (2) $S_{1,1}X_{1,2} + S_{1,2}X_{2,2} = X_{1,1}S_{1,2} + X_{1,2}S_{2,2}$, that is, $S_{1,1}X_{1,2} - X_{1,2}S_{2,2} = X_{1,1}S_{1,2} - S_{1,2}X_{2,2}$.

By Lemma 3.5, we may choose, without loss of generality, $X_{0,1} = 0$ and $X_{1,2} = 0$. And there exists $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \phi(S_{i,i}), i = 0, 1, 2$. It is therefore enough to find an operator $X_{0,2}$ satisfying

$$S_{0,0}X_{0,2} - X_{0,2}S_{2,2} = X_{0,0}S_{0,2} - S_{0,2}X_{2,2}.$$

Clearly, we have

$$\begin{aligned} (X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2(w)) &= X_{0,0}(m_{0,2}t_0^{(1)}(w)) - S_{0,2}(\phi(w)t_2(w)) \\ &= m_{0,2}(\phi(w)t_0(w))^{(1)} - m_{0,2}\phi(w)t^{(1)}(w) \\ &= m_{0,2}\phi^{(1)}(w)t_0(w). \end{aligned}$$

We therefore set $X_{0,2}$ be the operator: $X_{0,2}(t_2(w)) = m_{0,2}\phi^{(1)}(w)t_0^{(1)}(w)$.

To complete the proof by induction, we assume that we have the validity of the conclusion for $n = k$. Thus we assume the existence of a bounded linear operator $X = \langle\langle X_{i,j} \rangle\rangle$ such that $\langle\langle S_{i,j} \rangle\rangle \langle\langle X_{i,j} \rangle\rangle = \langle\langle X_{i,j} \rangle\rangle \langle\langle S_{i,j} \rangle\rangle$ where $X_{i,i} = \phi(S_{i,i})$ and $X_{i,i+1} = 0$. And there exists $l_{i,j}^r$ such that $X_{i,j}(t_j) =$

$\sum_{r=1}^{j-i-1} l_{i,j}^r \phi^{(j-k)} t_i^{(k)}$. To complete the inductive step, we only need to find the operator $X_{0,k}$ satisfying the following equation:

$$(3.3) \quad S_{0,0}X_{0,k} - X_{0,k}S_{k,k} = X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + \left(\sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=1}^{k-2} S_{0,i}X_{i,k} \right)$$

Note that the induction hypothesis ensures the existence of constants $c_{0,k}^s$ (depending on $m_{i,j}$) such that

$$(X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + \sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=1}^{k-2} S_{0,i}X_{i,k})(t_k) = \sum_{s=1}^{k-1} c_{0,k}^s \phi^{(s)} t_0^{(k-s-1)}.$$

Now, suppose that $X_{0,k}(t_k) = \sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} t_0^{(k-s)}$, where the constants $l_{0,k}^s$ are to be found. Then we must have

$$\begin{aligned} (S_{0,0}X_{0,k} - X_{0,k}S_{k,k})(t_k(w)) &= S_{0,0} \left(\sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} t_0^{(k-s)}(w) \right) - w \left(\sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} t_0^{(k-s)}(w) \right) \\ &= \sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} (w t_0^{(k-s)}(w) + (k-s)t^{(k-s-1)}(w)) - w \left(\sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} t_0^{(k-s)}(w) \right) \\ &= \sum_{s=1}^{k-1} l_{0,k}^s (k-s) \phi^{(s)} t^{(k-s-1)}(w) \\ &= \sum_{s=1}^{k-1} c_{0,k}^s \phi^{(s)} t_0^{(k-1-s)}(w) \end{aligned}$$

It follows that if we choose $l_{0,k}^s = \frac{c_{0,k}^s}{k-s}$, then $X_{0,k}$ with this choice of the constants validates equation (3.3). This completes the induction step.

In particular, when $\mu_{i,j}$ are all chosen to be 1, and then $m_{i,j} = -1$, i.e. $S_{i,j}(t_j) = -t_j^{(j-i-1)}$. In this case, $X_{0,k}(t_0) = -\sum_{s=1}^{k-1} \phi^{(s)} t_0^{(k-s)}$. And if $m_{i,j} = -1, i, j = 0, 1, \dots, n-1$, then by a same argument, we have that

$$(3.4) \quad X_{i,j}(t_j) = -\sum_{s=1}^{j-i-1} \phi^{(s)} t_i^{(j-i-s)}, i, j = 0, 1, \dots, n-1.$$

□

3.3. Proof of the main theorem.

Proof of Theorem 3.1. First, if “ $\Lambda(t) \geq 2$ ”, then the first conclusion of the theorem follows from Lemma 3.4. So, it remains for us to verify the second statement of the theorem, where $\Lambda(t) < 2$.

Let T and \tilde{T} be the operators representing t and \tilde{t} respectively. Recall that $S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}$, $\tilde{S}_{i,j}(t_j) = \tilde{m}_{i,j} t_i^{(j-i-1)}$. Up to similarity, we can assume that $m_{i,i+1} = \tilde{m}_{i,i+1}$. Then T and \tilde{T} have the following atomic decomposition:

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} \text{ and } \tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & \cdots & c_{0,n-1}S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & c_{1,n-1}S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & c_{n-2,n-1}S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}$$

Set $c_{i,j} = \frac{\tilde{m}_{i,j}}{m_{i,j}}$. Now it is enough to prove the Claim stated below.

Claim: If $T \sim \tilde{T}$, then $c_{i,j} = 1, i, j = 0, 1, \dots, n$.

Consider the following possibilities:

- (1) $\Lambda(t) \in [0, 1)$
- (2) $n = 3, \Lambda(t) \in [1, 2); n > 3, \Lambda(t) \in [1, \frac{4}{3})$
- (3) $n = 4, \Lambda(t) \in [\frac{4}{3}, 2); n > 4, \Lambda(t) \in [\frac{4}{3}, \frac{3}{2})$
- (4) $n = 5, \Lambda(t) \in [\frac{3}{2}, 2); n > 5, \Lambda(t) \in [\frac{3}{2}, \frac{8}{5})$

The proofs of the remaining cases are similar.

In the following, without loss of generality, we will always choose $m_{i,j} = -1, i, j = 0, 1, \dots, n-1$.

Case (1): By Lemma 2.2, we have

$$T = \tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & S_{n-1,n-1} \end{pmatrix}.$$

In this case, we clearly have $K_{t_i} = K_{s_i}$ and $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}, i = 0, 1, \dots, n-1$.

Case (2): By Lemma 3.2, we have

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & 0 & 0 \\ & S_{1,1} & S_{1,2} & S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & S_{n-1,n-1} \end{pmatrix}.$$

In the case, by Lemma 2.2, we first assume that $n = 3$. Then we have

$$(3.5) \quad \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}$$

By Lemma 3.5, $X_{0,1}$ and $X_{1,2}$ may be chosen to be zero. For the general case, we may also choose $X_{i,i+1}, i = 0, 1, \dots, n-1$ to be zero by repeating the same argument.

Now we have the following equality

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}.$$

And

$$\begin{aligned} S_{i,i+1}X_{i+1,i+1} &= X_{i,i}S_{i,i+1}, i = 0, 1, \\ S_{0,0}X_{0,2} + S_{0,2}X_{2,2} &= c_{0,2}X_{0,0}S_{0,2} + X_{0,2}S_{2,2}, \\ S_{0,0}X_{0,2} - X_{0,2}S_{2,2} &= c_{0,2}X_{0,0}S_{0,2} - S_{0,2}X_{2,2}. \end{aligned}$$

By

$$S_{i,i+1}X_{i+1,i+1} = X_{i,i}S_{i,i+1}, i = 0, 1,$$

and $\mathcal{A}'(S_{i,i}) \cong \mathcal{H}^\infty(\mathbb{D})$, by Lemma 3.5, we can find a holomorphic function $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i}t_i = \phi t_i$. Since $X_{i,i}$ is invertible, $\phi(S_{i,i})$ is also invertible. Note that

$$\begin{aligned} (c_{0,2}X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2) &= c_{0,2}X_{0,0}(-t_0^{(1)}) - S_{0,2}(\phi t_2) \\ &= \phi t_0^{(1)} - c_{0,2}(\phi t_0)^{(1)} \\ (3.6) \quad &= \phi t_0^{(1)} - c_{0,2}\phi t_0^{(1)} - c_{0,2}\phi^{(1)}t_0 \\ &= (1 - c_{0,2})\phi t_0^{(1)} - c_{0,2}\phi^{(1)}t_0 \\ &= (c_{0,2} - 1)S_{0,2}\phi(S_{2,2})(t_2) - c_{0,2}S_{0,1}S_{1,2}\phi^{(1)}(S_{2,2})(t_2). \end{aligned}$$

By Lemma 3.3, we have $c_{0,2}S_{0,1}S_{1,2}\phi^{(1)}(S_{2,2}) \in \text{Ran}\sigma_{S_{0,0},S_{2,2}}$. From (3.6), it follows that

$$(c_{0,2} - 1)S_{0,2}\phi(S_{2,2}) \in \text{ran}\sigma_{S_{0,0},S_{2,2}}.$$

By Lemma 3.2, $S_{0,2} \notin \text{ran}\sigma_{S_{0,0},S_{2,2}}$. Since $\phi(S_{2,2})$ is invertible and $\phi(S_{2,2}) \in \mathcal{A}'(S_{2,2})$, we have

$$S_{0,2}\phi(S_{2,2}) \notin \text{ran}\sigma_{S_{0,0},S_{2,2}}$$

it follows from Lemma 3.4. This shows that $c_{0,2} = 1$. For the general case, by the above argument and Lemma 2.2, we have

$$\tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & c_{1,3}S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & S_{n-2,n-2} & S_{n-2,n-1} & c_{n-2,n}S_{n-2,n} \\ & 0 & & & S_{n-1,n-1} & S_{n-1,n} \\ & & & & & S_{n,n} \end{pmatrix}.$$

Now suppose that we have proved Claim 1 for $n = k - 1$. Pick $X = \begin{pmatrix} X_{0,0} & 0 & \cdots & X_{0,k} \\ 0 & X_{1,1} & \cdots & X_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{k,k} \end{pmatrix}$ such that

$X\tilde{T} = TX$. Then it follows that

$$X_0((\tilde{S}_{i,j})_{i,j=0}^{k-1}) = ((S_{i,j})_{i,j=0}^{k-1})X_0, X_1((\tilde{S}_{i,j})_{i,j=1}^k) = ((S_{i,j})_{i,j=1}^k)X_1,$$

where

$$X_0 = \begin{pmatrix} X_{0,0} & 0 & \cdots & X_{0,k-1} \\ 0 & X_{1,1} & \cdots & X_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k-1,k-1} \end{pmatrix}, \quad X_1 = \begin{pmatrix} X_{1,1} & 0 & \cdots & X_{1,k} \\ 0 & X_{2,2} & \cdots & X_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k,k} \end{pmatrix}.$$

Since X is invertible, X_0 and X_1 are both invertible. By the induction hypothesis $c_{i,i+2} = 1, i = 0, 1, \dots, n - 3$.

Case (3) and Case (4): By Lemma 2.2, $\tilde{S} = (\tilde{S}_{i,j}), \tilde{S}_{i,j} = 0, j - i \geq 4$ and $\tilde{S} = (\tilde{S}_{i,j}), \tilde{S}_{i,j} = 0, j - i \geq 5$. Following the proof given above, by Lemma 2.2, we only need to consider the case of $n = 4$ and $n = 5$. For case 3, we only consider $n = 4$ and the other cases would follow by induction. In this case, we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} \\ 0 & S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & 0 & S_{3,3} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} \\ 0 & X_{1,1} & 0 & X_{1,3} \\ 0 & 0 & X_{2,2} & 0 \\ 0 & 0 & 0 & X_{3,3} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} \\ 0 & X_{1,1} & 0 & X_{1,3} \\ 0 & 0 & X_{2,2} & 0 \\ 0 & 0 & 0 & X_{3,3} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & c_{0,3}S_{0,3} \\ 0 & S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & 0 & S_{3,3} \end{pmatrix}.$$

It follows that $\begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}$ and $\begin{pmatrix} X_{1,1} & 0 & X_{1,3} \\ 0 & X_{2,2} & 0 \\ 0 & 0 & X_{3,3} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & S_{3,3} \end{pmatrix}$. By (3.4) in the proof of Lemma 3.6, we have $X_{0,2}$ and $X_{1,3}$ can be chosen as $S_{0,2}\phi^{(1)}(S_{2,2})$ and $S_{1,3}\phi^{(1)}(S_{3,3})$. Note that

$$S_{0,0}X_{0,3} + S_{0,1}X_{1,3} + S_{0,3}X_{3,3} = c_{0,3}X_{0,0}S_{0,3} + X_{0,2}S_{2,3} + X_{0,3}S_{3,3}.$$

Then

$$(3.7) \quad S_{0,0}X_{0,3} - X_{0,3}S_{3,3} = (c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3}) + X_{0,2}S_{2,3} - S_{0,1}X_{1,3}.$$

And

$$\begin{aligned} X_{0,2}S_{2,3} - S_{0,1}X_{1,3} &= S_{0,2}\phi^{(1)}(S_{2,2})S_{2,3} - S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) \\ &= S_{0,2}S_{2,3}\phi^{(1)}(S_{3,3}) - S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) \\ &= (S_{0,2}S_{2,3} - S_{0,1}S_{1,3})\phi^{(1)}(S_{3,3}) \\ &= 0. \end{aligned}$$

So we only need to consider

$$S_{0,0}X_{0,3} - X_{0,3}S_{3,3} = c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3}.$$

Since

$$\begin{aligned} (c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3})(t_3) &= c_{0,3}X_{0,0}S_{0,3}(t_3) - S_{0,3}(\phi t_3) \\ &= c_{0,3}X_{0,0}(-t_0^{(2)}) - \phi S_{0,3}(t_3) \\ &= -c_{0,3}(\phi t_0)^{(2)} + \phi t_0^{(2)} \\ &= (1 - c_{0,3})\phi t_0^{(2)} - 2c_{0,3}\phi^{(1)}t_0^{(1)} - c_{0,3}\phi^{(2)}t_0, \end{aligned}$$

we obtain

$$c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3} = (c_{0,3} - 1)S_{0,3}\phi(S_{3,3}) + 2c_{0,3}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + c_{0,3}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}).$$

By Lemma 3.3 and (3.7), we have

$$2c_{0,3}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + c_{0,3}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) \in \text{Ran}\sigma_{S_{0,0},S_{3,3}}.$$

Since $\phi(S_{3,3})$ is invertible, we deduce that

$$(c_{0,3} - 1)S_{0,3} \in \text{ran}\sigma_{S_{0,0},S_{3,3}}.$$

Note that $S_{0,3} \notin \text{ran}\sigma_{S_{0,0},S_{3,3}}$, we have $c_{0,3} = 1$. For case 4, when $n = 5$, the commutator

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & & & S_{3,3} & S_{3,4} \\ & 0 & & & S_{4,4} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & & & X_{3,3} & 0 \\ & 0 & & & X_{4,4} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & & & X_{3,3} & 0 \\ & 0 & & & X_{4,4} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & c_{0,4}S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & & & S_{3,3} & S_{3,4} \\ & 0 & & & S_{4,4} \end{pmatrix}$$

Therefore $(X_{ij})_{4 \times 4}$, $i, j = 0, 1, 2, 3$ commutes with $(S_{i,j})_{4 \times 4}$, $i, j = 0, 1, 2, 3$ and $(X_{ij})_{4 \times 4}$, $i, j = 1, 2, 3, 4$ commutes with $(S_{i,j})_{4 \times 4}$, $i, j = 1, 2, 3, 4$. Then we can find $X_{i,j}$, $(i, j) \neq (0, 4)$ from Lemma 3.5. We also have

$$(3.8) \quad S_{0,0}X_{0,4} - X_{0,4}S_{4,4} = (c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4}) + (X_{0,2}S_{2,4} + X_{0,3}S_{3,4}) - (S_{0,1}X_{1,4} + S_{0,2}X_{2,4}).$$

By Lemma 3.5, we have

$$\begin{aligned} X_{0,2}S_{2,4} - S_{0,2}X_{2,4} &= S_{0,2}\phi^{(1)}(S_{2,2})S_{2,4} - S_{0,2}S_{2,4}\phi^{(1)}(S_{4,4}) \\ &= S_{0,2}(S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{2,4}\phi^{(1)}(S_{4,4})) - S_{0,2}S_{2,4}\phi^{(1)}(S_{4,4}) \\ &= S_{0,2}S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}). \end{aligned}$$

By (3.4) in the proof of Lemma 3.6, we have

$$X_{0,3} = S_{0,2}S_{2,3}\phi^{(2)}(S_{3,3}) + S_{0,3}\phi^{(1)}(S_{3,3}),$$

$$X_{1,4} = S_{1,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{1,4}\phi^{(1)}(S_{4,4}).$$

Note that $S_{0,2}S_{2,3} = S_{0,1}S_{1,3}$ and $S_{0,3}S_{3,4} = S_{0,1}S_{1,4}$, we also have

$$\begin{aligned} &X_{0,3}S_{3,4} - S_{0,1}X_{1,4} \\ &= (S_{0,2}S_{2,3}\phi^{(2)}(S_{3,3}) + S_{0,3}\phi^{(1)}(S_{3,3}))S_{3,4} - S_{0,1}(S_{1,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{1,4}\phi^{(1)}(S_{4,4})) \\ &= S_{0,2}S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{0,3}S_{3,4}\phi^{(1)}(S_{4,4}) - S_{0,1}S_{1,3}S_{3,4}\phi^{(2)}(S_{4,4}) - S_{0,1}S_{1,4}\phi^{(1)}(S_{4,4}) \\ &= 0. \end{aligned}$$

Since

$$\begin{aligned} (c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4})(t_4) &= c_{0,4}X_{0,0}S_{0,4}(t_4) - S_{0,4}(\phi t_4) \\ &= c_{0,4}X_{0,0}(-t_0^{(3)}) - \phi S_{0,4}(t_4) \\ &= -c_{0,4}(\phi t_0)^{(3)} + \phi t_0^{(3)} \\ &= (1 - c_{0,4})\phi t_0^{(3)} - 3c_{0,4}\phi^{(2)}t_0^{(1)} - 3c_{0,4}\phi^{(1)}t_0^{(2)} - c_{0,4}\phi^{(3)}t_0, \end{aligned}$$

and we also have

$$\begin{aligned} c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4} &= (c_{0,4} - 1)S_{0,4}\phi(S_{4,4}) + 3c_{0,4}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) \\ &\quad + 3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(3)}(S_{3,3}). \end{aligned}$$

By Lemma 3.3 and (3.8), we obtain

$$\begin{aligned} 3c_{0,4}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + 3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(3)}(S_{3,3}) &\in \text{ran } \sigma_{S_{0,0}, S_{4,4}}, \\ S_{0,2}S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}) &\in \text{Ran } \sigma_{S_{0,0}, S_{4,4}}. \end{aligned}$$

Then it follows that

$$(c_{0,4} - 1)S_{0,4}\phi(S_{4,4}) \in \text{ran } \sigma_{S_{0,0}, S_{4,4}}.$$

Note that $\phi(S_{4,4})$ is invertible, therefore

$$(c_{0,4} - 1)S_{0,4} \in \text{ran } \sigma_{S_{0,0}, S_{4,4}}.$$

Since $S_{0,4} \notin \text{ran } \sigma_{S_{0,0}, S_{4,4}}$, it follows that $c_{0,4} = 1$.

The proof in all the remaining cases are similar and therefore the Claim is verified. \square

4. APPLICATIONS

We give three different applications of our results. First of these shows that the topological and algebraic K -groups defined in our context must coincide. Secondly, we show that our techniques apply to a slightly larger class of operators than the quasi-homogeneous ones that we have discussed in this paper. Finally, we show that the Halmos' question on similarity has an affirmative answer for quasi-homogeneous operators. We begin with some preliminaries on K -groups.

4.1. Preliminaries. Let $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ be a holomorphic curve. Recall that the commutant $\mathcal{A}'(E_t)$ of such a holomorphic curve t is defined to be

$$\mathcal{A}'(E_t) = \{A \in \mathcal{L}(\mathcal{H}) : At(w) \subseteq t(w), w \in \Omega.\}$$

Definition 4.1. For a holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$, the Jacobson radical $\text{Rad } \mathcal{A}'(E_t)$ of $\mathcal{A}'(E_t)$ is defined to be

$$\{S \in \mathcal{A}'(E_t) \mid \sigma_{\mathcal{A}'(E_t)}(SA) = 0, A \in \mathcal{A}'(E_t)\},$$

where $\sigma_{\mathcal{A}'(E_t)}(SA)$ denotes the spectrum of SA in the algebra $\mathcal{A}'(E_t)$.

The discussion below follows closely the paper [17] of the first two authors. In particular, Lemma 4.2 and Lemma 4.3 are proved there.

Lemma 4.2. ([17, Theorem 1.2]) Let $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ be a holomorphic curve, and $P \in \mathcal{A}'(E_t)$ be an idempotent, then $Pt : \Omega \rightarrow \text{Gr}(m, P\mathcal{H})$ is again a holomorphic curve, where $m = \dim \text{ran } P(t(w))$ for $w \in \Omega$. The idempotent P is minimal if and only if Pt is strongly irreducible.

Lemma 4.3. ([17, Theorem 1.3]) For a holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$, the following statements are equivalent.

- (1) There exists m minimal idempotents $P_1, P_2, \dots, P_m \in \mathcal{A}'(E_t)$ such that $P_i P_j = 0$ and $\sum_{i=1}^m P_i = I_{\mathcal{H}}$.
- (2) There exists an invertible operator $X \in \mathcal{A}'(E_t)$ such that Xt can be written as orthogonal direct sum of m strongly irreducible holomorphic curves.

Definition 4.4. A holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$ is said to have a finite decomposition if it meets one of the equivalent conditions given in Lemma 4.3.

Suppose $\{P_1, P_2, \dots, P_m\}$ and $\{Q_1, Q_2, \dots, Q_n\}$ are two distinct decompositions of t . If $m = n$, there exists a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for some invertible operator X in $\mathcal{A}'(E_t)$, $1 \leq i \leq n$, then we say that t (or E_t) has a unique decomposition up to similarity.

For a holomorphic curve, $f : \Omega \rightarrow Gr(n, \mathcal{H})$, let $M_k(\mathcal{A}'(E_t))$ be the collection of $k \times k$ matrices with entries from $\mathcal{A}'(E_t)$. Let

$$M_\infty(\mathcal{A}'(E_t)) = \bigcup_{k=1}^{\infty} M_k(\mathcal{A}'(E_t)),$$

and $\text{Proj}(M_k(\mathcal{A}'(E_t)))$ be the algebraic equivalence classes of idempotents in $M_\infty(\mathcal{A}'(E_t))$. If p, q are idempotents in $\text{Proj}(\mathcal{A}'(E_t))$, then say that $p \sim_{st} q$ if $p \oplus r \sim_a q \oplus r$ for some idempotent r in $\text{Proj}(\mathcal{A}'(E_t))$. The relation \sim_{st} is known as stable equivalence.

Let X be a compact Hausdorff space, and $\xi = (E, \pi, X)$ be a (topological) vector bundle. A well-known theorem due to R. G. Swan (cf. [31]) says that a vector bundle $\xi = (E, \pi, X)$ is a direct summand of the trivial bundle, that is,

$$\xi \oplus \eta \cong (X \times \mathbb{C}^n, \pi, X)$$

for some vector bundle $\eta = (F, \rho, X)$.

Swan's Theorem relates the geometric notion of a vector bundle to the algebraic notion of a K_0 group which we now describe briefly.

Following the usual conventions, let $\text{Vect}(X)$ be the set of all isomorphism classes $\bar{\xi}$ of vector bundles ξ over X . Addition and multiplication are defined on $\text{Vect}(X)$ by the rule

$$\bar{\xi} + \bar{\eta} = \overline{\xi \oplus \eta}, \bar{\xi} \bar{\eta} = \overline{\xi \otimes \eta}.$$

These operations are well defined. Thus $(\text{Vect}(X), +)$ is an Abelian semi-group and $K^0(X)$ is defined as the Grothendieck group of $(\text{Vect}(X), +)$ (see [28] for more details). Swan's theorem is the main ingredient in showing that the topological K -group $K^0(X)$ is isomorphic to the algebraic K_0 -group $K_0(C(X))$.

For any projection $p \in P(M_\infty(C(X)))$, suppose that $p \in M_n(C(X))$. From this p , one may construct a vector bundle on X :

$$E(p) := \{(x, v) \in X \times \mathbb{C}^n : v \in p(x)(\mathbb{C}^n)\},$$

with the fiber $E_x(p) = p(x)(\mathbb{C}^n)$. Define an additive map $\Gamma : \text{Proj}(\mathcal{A}'(E_t)) \rightarrow \text{Vect}(X)$ as follows:

$$\Gamma([p]_0) = \bar{\xi}_p, [p]_0 \in \text{Proj}(\mathcal{A}'(E_t)).$$

Then Γ is an isomorphism.

First, we show Γ is injective. If $\Gamma([p]_0) = \bar{\xi}_p = \bar{\xi}_q = \Gamma([q]_0)$, and $p \in P(M_n(C(X))), q \in P(M_m(C(X)))$ then there exists an isomorphism

$$\sigma : \bar{\xi}_p \rightarrow \bar{\xi}_q,$$

where $\sigma(p(x)\mathbb{C}^n) \cong (q(x)\mathbb{C}^m)$. So we have $\text{Tr}(p(x)) = \text{Tr}(q(x))$, where Tr is the trace of $M_\infty(C(X))$. Then we can find $v_x \in M_{m,n}(C(X))$ such that

$$v_x^* v_x = p(x), v_x v_x^* = q(x).$$

That means $[p]_0 = [q]_0$. So Γ is injective.

Next, we show that Γ is surjective. By Swan's theorem, for any vector bundle $\xi = (E, \pi, X)$, there exists a positive integer n and another vector bundle $\eta(F, \rho, X)$ such that

$$\xi \oplus \eta = (X \times \mathbb{C}^n, \pi, X).$$

The we can assume that $E_x \oplus F_x = \mathbb{C}^n$. Set $p(x)$ be the projection from \mathbb{C}^n onto E_x . Then $p : X \rightarrow M_n(\mathbb{C})$ is continuous and $p \in P(M_n(C(X)))$, $\xi := \xi_p$. Then we can see that Γ is also a surjective. So Γ is an isomorphism.

Thus if $\xi \oplus \eta$ is a trivial bundle, then there exists $p \in M_\infty(C(X))$ such that $\xi = \xi_p, \eta = \xi_{I-p}$. Now, if there exists another vector bundle η' such that $\xi \oplus \eta'$ is also isomorphic to a trivial bundle, then

there must exist a projection $p' \in M_\infty(C(X))$ such that $\xi = \xi_{p'}, \eta = \xi_{I-p'}$. Consequently, $[p]_0 = [p']_0$, $[1-p]_0 = [1-p']_0$ and we see that $\eta' \cong \eta$. So there is a unique vector bundle η such that

$$\xi \oplus \eta \cong (X \times \mathbb{C}^n, \pi, X).$$

4.2. Unique decomposition. None of what we have said so far applies to holomorphic vector bundles over an open subset of \mathbb{C} since they are already trivial by Grauert's theorem. However, the study of holomorphic vector bundles over an open subset of \mathbb{C} is central to operator theory. In the context of operator theory, as shown in the foundational paper of Cowen and Douglas [2], the vector bundles of interest are equipped with a Hermitian structure inherited from a fixed inner product of some Hilbert space \mathcal{H} . This makes it possible to ask questions about their equivalence under a unitary or an invertible linear transformation of \mathcal{H} . In the paper [2], questions regarding unitary equivalence were dealt with quite successfully while equivalence under an invertible linear transformation remains somewhat of a mystery to date. However, we can ask if the uniqueness of the summand, which was a consequence of Swan's theorem, remains valid in the context of Cowen-Douglas operators.

Question. Let $t : \Omega \rightarrow Gr(n, \mathcal{H})$ be a Hermitian holomorphic curve and the vector bundle E_r be a direct summand of E_t for some other holomorphic curve $r : \Omega \rightarrow Gr(n, \mathcal{H})$. Does there a unique sub-bundle of E_t , up to similarity, such that $E_r \oplus E_s = E_t$? Here the uniqueness is meant to be in the sense of Definition 4.4

It was shown in [18] that an operator in the Cowen-Douglas class $B_n(\Omega)$ admits a unique decomposition. So, the answer to the question raised above is affirmative. However, here we give a different proof for quasi-homogeneous operators which is much more transparent. For our proof, we will need the following lemma.

Lemma 4.5. *Let E_t be a quasi-homogeneous bundle. Then $\mathcal{A}'(E_t)/\text{Rad}(\mathcal{A}'(E_t))$ is commutative.*

Proof. Let

$$\mathcal{S} = \{Y : \sigma(Y) = 0, Y \in \mathcal{A}'(E_t)\}.$$

Claim 1: \mathcal{S} is an ideal of the algebra $\mathcal{A}'(E_t)$.

By Lemma 2.6, Y is upper-triangular if $Y \in \mathcal{S}$. Since the spectrum $\sigma(Y)$ of Y is $\{0\}$, the operator Y must be of the form

$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ 0 & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & Y_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

and it follows that each quasi-nilpotent element in the commutant of the holomorphic curve t of rank one is zero. Using Lemma 2.6 again, each element $X \in \mathcal{A}'(E_t)$ is upper-triangular. Thus, $\sigma(XY) = \sigma(YX) = 0$. This completes the proof of Claim 1 and $\mathcal{S} = \text{Rad}(\mathcal{A}'(E_t))$.

Claim 2: $\mathcal{A}'(E_t)/\text{Rad}(\mathcal{A}'(E_t))$ is commutative.

Note that if $X \in \mathcal{A}'(E_t)$ is (block) nilpotent, then $X \in \mathcal{S}$. A simple computation shows that $\mathcal{A}'(E_t)/\text{Rad}(\mathcal{A}'(E_t))$ is commutative. \square

Theorem 4.6. *For any quasi-homogeneous holomorphic curve t with atoms t_i , $0 \leq i \leq n-1$, we have that*

- (1) E_t has no non-trivial sub-bundle whenever $\Lambda(t) < 2$, and
- (2) if $\Lambda(t) \geq 2$, then for any sub-bundle E_r of E_t , there exists a unique sub-bundle E_s , up to equivalence under an invertible map, such that $E_r \oplus E_s$ is similar to E_t .

For any holomorphic curve t , we let t^n denote the n -fold direct sum of t . For any two natural numbers n and m , let E_r and E_s be the sub-bundles of E_{t^n} and E_{t^m} , respectively. If $m > n$, then both E_r and E_s can be regarded as a sub-bundle of E_{t^m} .

Two holomorphic Hermitian vector bundles E_r and E_s are said to be similar if there exist an invertible operator $X \in \mathcal{A}'(E_r)$ such that $XE_r = E_s$. Analogous to the definition of $\text{Vect}(X)$, we let $\text{Vect}^0(E_t)$ be the set of equivalence classes $\overline{E_s}$ of the sub-bundles E_s of E_t^n , $n = 1, 2, \dots$. An addition on $\text{Vect}^0(E_t)$ is defined as follows, namely,

$$\overline{E_r} + \overline{E_s} = \overline{E_r \oplus E_s},$$

where E_r and E_s are both sub-bundles of E_t . Now, the group $K^0(E_t)$ is the Grothendieck group of $(\text{Vect}^0(E_t), +)$. In this notation, we have the following theorem.

Theorem 4.7. $K^0(E_t) \cong K_0(\mathcal{A}'(E_t))$.

The proof of this theorem is split into a number of lemmas which are stated and proved below.

Lemma 4.8. *Let E_t be a quasi-homogeneous bundle. Then*

$$\text{Vect}(\mathcal{A}'(E_t)) \cong \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t)).$$

Proof. Note that $M_n(\mathcal{A}'(E_t)) \cong \mathcal{A}'(\bigoplus^n E_t)$. Let $p \in M_n(\mathcal{A}'(E_t))$ be an idempotent. Define a map $\sigma : \text{Vect}(\mathcal{A}'(E_t)) \rightarrow \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t))$ as the following:

$$\sigma[P] = [\pi(P)],$$

where $\pi : \mathcal{A}'(E_t) \rightarrow \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t))$.

Claim σ is well defined and it is an isomorphism.

If $[p] = [q]$, where $p \in M_n(\mathcal{A}'(E_t))$ and $q \in M_m(\mathcal{A}'(E_t))$ are both idempotents, then there exists $k \geq \max\{m, n\}$ and an invertible element $u \in M_k(\mathcal{A}'(E_t))$ such that

$$u(p \oplus 0^{k-n})u^{-1} = q \oplus 0^{k-m}.$$

Thus we have

$$\pi(u)\pi(p \oplus 0^{k-n})\pi(u)^{-1} = \pi(u(p \oplus 0^{k-n})u^{-1}) = \pi(q \oplus 0^{k-m}).$$

That means $[\pi(p)] = [\pi(q)]$, and σ is well defined.

Then we would prove that σ is injective. In fact, if $p \in M_n(\mathcal{A}'(E_t))$ and $q \in M_m(\mathcal{A}'(E_t))$ are idempotents with

$$\sigma[p] = [\pi(p)] = [\pi(q)] = \sigma[q],$$

then we can find $k \geq \max\{m, n\}$ and an invertible element $\pi(u) \in M_k(\mathcal{A}'(E_t))/\text{Rad}(M_k(\mathcal{A}'(E_t)))$ such that

$$\pi(u)(\pi(p \oplus 0^{k-n}))\pi(u)^{-1} = \pi(q \oplus 0^{k-m}).$$

Since $\pi(u)$ is invertible, there exists $\pi(s) \in \text{Rad}(M_k(\mathcal{A}'(E_t)))$ such that $\pi(u)^{-1} = \pi(s)$. Then we have

$$us = I - R_1, su = I - R_2,$$

where $R_1, R_2 \in \text{Rad}(M_k(\mathcal{A}'(E_t)))$. Since $\sigma(R_1) = \sigma(R_2) = \{0\}$, then us, su are both invertible. Therefore, u is invertible and thus

$$\pi(u(p \oplus 0^{k-n})u^{-1}) = \pi(u)(\pi(p \oplus 0^{k-n}))\pi(u)^{-1} = \pi(q \oplus 0^{k-m}).$$

Thus,

$$u(p \oplus 0^{k-n})u^{-1} = q \oplus 0^{k-m} + R$$

for some $R \in \text{Rad}(M_k(\mathcal{A}'(E_t)))$. Let $W_1 = 2(q \oplus 0^{k-m}) - I$. Since $\sigma(Q \oplus 0^{k-m}) \subseteq \{0, 1\}$, then W_1 is invertible. Since we have $R \in \text{Rad}(M_k(\mathcal{A}'(E_t)))$ and $W_1^{-1} \in M_k(\mathcal{A}'(E_t))$, then $RW_1^{-1} \in \text{Rad}(M_k(\mathcal{A}'(E_t)))$, so $I + RW_1^{-1}$ is invertible. Set

$$W = 2(q \oplus 0^{k-m}) - I + R = W_1 + R = (I + RW_1^{-1})W_1.$$

and W is invertible. Since $p \oplus 0^{(k-n)}$ is an idempotent, then $u(p \oplus 0^{(k-n)})u^{-1}$ is an idempotent, then $(q \oplus 0^{(k-m)}) + R$ is an idempotent. Thus,

$$(q \oplus 0^{(k-m)})^2 + (q \oplus 0^{(k-m)})R = R(q \oplus 0^{(k-m)}) + R^2 = (q \oplus 0^{(k-m)}) + R.$$

Since $q \oplus 0^{(k-m)}$ is an idempotent, then

$$(q \oplus 0^{(k-m)})R + R(q \oplus 0^{(k-m)}) + R^2 = R.$$

So we have

$$\begin{aligned} W((q \oplus 0^{(k-m)}) + R) &= (q \oplus 0^{(k-m)}) + R(q \oplus 0^{(k-m)}) + 2(q \oplus 0^{(k-m)})R - R + R^2 \\ &= (q \oplus 0^{(k-m)}) + (q \oplus 0^{(k-m)})R \\ &= (q \oplus 0^{(k-m)})W. \end{aligned}$$

And

$$u(p \oplus 0^{(k-n)})u^{-1} = (q \oplus 0^{(k-m)}) + R = W^1(q \oplus 0^{(k-m)})W.$$

It follows that $p \sim_a q$, and σ is injective. At last, we would show that σ is surjective. For each $[\pi(p)] \in \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t))$ with $\pi(p) \in M_n(\mathcal{A}'(E_t))/\text{Rad}(M_n(\mathcal{A}'(E_t)))$, $p \in M_n(\mathcal{A}'(E_t))$ and $\pi^2(p) = \pi(p)$, we have

$$p^2 - p = R_0, R_0 \in \text{Rad}(M_n(\mathcal{A}'(E_t))).$$

Note that $p = B + R$, where $B \in M_n(\mathcal{A}'(E_t))$ is a block-diagonal matrix over \mathbb{C} , $R \in \text{Rad}(M_n(\mathcal{A}'(E_t)))$. Then $\pi(p) = \pi(B)$ and

$$R_0 = p^2 - p = (B + R)^2 - (B + R) = B^2 - B + (BR + RB + R^2 - R).$$

Since $\text{Rad}(M_n(\mathcal{A}'(E_t)))$ is an ideal of $M_n(\mathcal{A}'(E_t))$, then we have

$$B^2 - B \in \text{Rad}(M_n(\mathcal{A}'(E_t))).$$

Since B is a block-diagonal matrix, then we have B is also an idempotent. Then we have

$$\sigma([B]) = [\pi(p)].$$

That means σ is also a surjective. And we also can see that σ is homomorphism. Then σ is an isomorphism and

$$\text{Vect}(\mathcal{A}'(E_t)) \cong \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t)).$$

□

We need two more lemmas, which have been already proved in [17], we reproduce them below.

Lemma 4.9. ([17, Lemma 2.10]) *For any holomorphic curve $t : \Omega \rightarrow \text{Gr}(n, \mathcal{H})$, the following statements are equivalent.*

- (1) *Assume that \mathcal{H} has the decomposition $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i^{(n_i)}$, $1 \leq i \leq k$. The holomorphic curve t is similar to $\bigoplus_{i=1}^k (P_i e)^{(n_i)}$, $k, n_i < \infty$, where $P_i : \mathcal{H} \rightarrow \mathcal{H}_i$ are idempotents such that $P_i e$ is indecomposable, $P_i e \not\sim P_j e$ for $i \neq j$ and $t^{(\ell)}$ admits a finite unique decomposition, up to similarity, for $\ell \in \mathbb{N}$.*
- (2) *The algebra $\text{Vect}(\mathcal{A}'(t))$ is isomorphic to $\mathbb{N}^{(k)}$ via h , which maps $[I]$ to (n_1, n_2, \dots, n_k) , that is, $h([I]) = n_1 e_1 + n_2 e_2 + \dots + n_k e_k$, where I is the unit of $\mathcal{A}'(t)$, $0 \neq n_i \in \mathbb{N}$, and e_i are the generators of $\mathbb{N}^{(k)}$, $i = 1, 2, \dots, k$.*

Lemma 4.10. ([17, Lemma 2.14])

$$\text{Vect}(\mathcal{H}^\infty(\mathbb{D})) \cong \mathbb{N}, K_0(\mathcal{H}^\infty(\mathbb{D})) \cong \mathbb{Z}.$$

Proposition 4.11. *Let E_t and $E_{\tilde{t}}$ be two quasi-homogeneous bundles with matchable bundles $\{E_{t_i}\}_{i=0}^{n-1}$ and $\{E_{s_i}\}_{i=0}^{n-1}$ respectively. If $\Lambda(t) < 2$, then E_t and $E_{\tilde{t}}$ are similarity equivalent if and only if*

$$K_0(\mathcal{A}'(E_t \oplus E_{\tilde{t}})) \cong \mathbb{Z}.$$

If $\Lambda(t) \geq 2$, then E_t and $E_{\tilde{t}}$ are similarity equivalent if and only if

$$K_0(\mathcal{A}'(E_t \oplus E_{\tilde{t}})) \cong \mathbb{Z}^n.$$

Proof. Suppose that $\Lambda(t) < 2$. Let

$$S = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & S_{n-1,n-1} & S_{n-1,n} \\ & & & & S_{n,n} \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & \cdots & X_{0,n-1} \\ & X_{1,1} & X_{1,2} & \cdots & X_{1,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & X_{n-1,n-1} & X_{n-1,n} \\ & & & & X_{n,n} \end{pmatrix}.$$

Claim 1: If $XS = SX$, then we have $X_{i,i} = X_{j,j}$, for any $i \neq j$.

In fact, for any $i = 0, 1, \dots, n-1$, we have

$$S_{i,i}X_{i,i+1} + S_{i,i+1}X_{i+1,i+1} = X_{i,i}S_{i,i+1} + X_{i,i+1}S_{i+1,i+1},$$

and

$$S_{i,i}X_{i,i+1} - X_{i,i+1}S_{i+1,i+1} = X_{i,i}S_{i,i+1} - S_{i,i+1}X_{i+1,i+1} = 0.$$

Since $X_{i,i} \in \mathcal{A}'(E_{t_i})$ and each E_{t_i} induces a Hilbert functional space \mathcal{H}_i with reproducing kernel $\frac{1}{(1-z\bar{w})^{\lambda_i}}$, then we have $\mathcal{A}'(E_{t_i}) \cong \mathcal{H}^\infty(\mathbb{D})$. Then there exists $\phi_{i,i} \in \mathcal{H}^\infty(\mathbb{D})$ such that

$$X_{i,i} = \phi_{i,i}(S_{i,i}), i = 0, 1, \dots, n-1.$$

Thus, we have

$$\phi_{i,i}(S_{i,i})S_{i,i+1} - S_{i,i+1}\phi_{i+1,i+1}(S_{i+1,i+1}) = 0.$$

Since $S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}$, then

$$S_{i,i+1}(\phi_{i,i} - \phi_{i+1,i+1})(S_{i+1,i+1}) = 0.$$

Note that $S_{i,i+1}$ has a dense range, then we can set

$$\phi_{i,i} = \phi, i = 0, 1, \dots, n-1.$$

Claim 2: $\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D})$.

Recall that $\text{Rad}\mathcal{A}'(E_t) = \{S \in \mathcal{A}'(E_t) | \sigma_{\mathcal{A}'(E_t)}(SS') = 0, S' \in \mathcal{A}'(E_t)\}$. Any $X \in \mathcal{A}'(E_t)$ is upper triangular by Lemma 2.6 and $\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t)$ is commutative by Lemma 4.5. Therefore if Y is in $\text{Rad}\mathcal{A}'(E_t)$, then we have

$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & Y_{n-1,n} \\ & & & & 0 \end{pmatrix}.$$

Define a map $\Gamma : \mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t) \rightarrow \mathcal{H}^\infty(\mathbb{D})$ by the rule:

$$\Gamma([X]) = \phi, \text{ where } X = ((X_{i,j}))_{n \times n}, X_{i,i} = \phi(S_{i,i}).$$

Obviously, Γ is well defined and if $\Gamma([X]) = 0$, then $\phi = 0$. Then $X_{i,i} = 0$, it follows that $X \in \text{Rad}\mathcal{A}'(E_t)$ and $[X] = 0$. So Γ is injective.

For any $\phi \in \mathcal{H}^\infty(\mathbb{D})$, set $X_{i,i} = \phi(S_{i,i}), i = 0, 1, 2, \dots, n-1$. By Lemma 3.6, we can construct the operators $X_{i,j}, j \neq i$ such that $X := (X_{i,j})_{n \times n} \in \mathcal{A}'(E_t)$. That means Γ is surjective. Then Γ is an isomorphism and

$$\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D}).$$

By Lemma 4.9 and Lemma 4.10, we have

$$\text{Vect}(\mathcal{A}'(E_t)) \cong \mathbb{N}, K_0(\mathcal{A}'(E_t)) \cong \mathbb{Z}.$$

By Lemma 4.9, we have E_t has a unique finite decomposition up to similarity. Similarly, $E_{\tilde{t}}$ also has a unique finite decomposition up to similarity.

If $E_t \sim E_{\tilde{t}}$, then $(t \oplus \tilde{t}) \sim t^{(2)}$. So we have

$$\text{Vect}(\mathcal{A}'(t \oplus \tilde{t})) \cong \text{Vect}(\mathcal{A}'(t^{(2)})) \cong \text{Vect}M_2(\mathcal{A}'(t)) \cong \mathbb{N}$$

and

$$K_0(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{Z}.$$

On the other hand, Note that t and \tilde{t} are both strongly irreducible. If $K_0(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{Z}$ and $\text{Vect}(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{N}$, then by Lemma 4.9, we have $t \sim \tilde{t}$, otherwise we will have

$$\text{Vect}(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{N}^2.$$

This is a contradiction. □

Proof of Theorem 4.6. When $\Lambda(t) < 2$, by Lemma 3.2, we have E_t is strongly irreducible. So there exists no non-trivial idempotent in $\mathcal{A}'(E_t)$, which is the same as saying that the vector bundle E_t has no non-trivial sub-bundle.

When $\Lambda(t) \geq 2$, by Lemma 3.2, we have

$$E_r \sim E_{t_0} \oplus E_{t_1} \oplus \cdots \oplus E_{t_{n-1}}.$$

Since $\mathcal{A}'(E_{t_i}) \cong \mathcal{H}^\infty(\mathbb{D})$, we have

$$\mathcal{A}'(E_r) \cong \mathcal{H}^\infty(\mathbb{D})^{(n)},$$

and by Lemma 4.10,

$$\text{Vect}(\mathcal{A}'(E_r)) \cong \mathbb{N}^{(n)}, K_0(\mathcal{A}'(E_r)) \cong \mathbb{Z}^{(n)}.$$

Then by Lemma 4.9, we have E_t has a unique finite decomposition up to similarity. Then for any non-trivial reducible sub-bundle of E_r denoted by E_r , with

$$\mathcal{H}_r = \text{Span}_{w \in \Omega} \{E_r(w)\}.$$

Let P_t be the projection from \mathcal{H} to \mathcal{H}_t . Then

$$E_t \sim E_r \oplus (E_t \ominus E_r) = P_r E_t \oplus (I - P_r) E_t.$$

Let

$$P_{t_i} : \mathcal{H} \rightarrow \mathcal{H}_i := \text{Span}_{\lambda \in \Omega} \{E_{t_i}(w)\}, i = 0, 1, \dots, n-1$$

be projections in $\mathcal{A}'(E_r)$. Then there exists $t_{k_i}, i = 0, 1, \dots, s$ such that

$$P \sim \bigoplus_{i=0}^s P_{t_{k_i}}.$$

Then it follows that

$$E_r \sim \bigoplus_{i=0}^s P_{t_{k_i}} E_t \sim \bigoplus_{i=0}^s E_{t_{k_i}},$$

namely, there exists an invertible operator X such that $E_r = X(\bigoplus_{i=0}^s E_{t_{k_i}})$. Suppose that

$$\bigoplus_{i=0}^{n-1} E_{t_i} = (\bigoplus_{i=0}^s E_{t_{k_i}}) \oplus (\bigoplus_{i=0}^{n-s} E_{t_{l_i}}).$$

Set

$$E_s = X(\bigoplus_{i=0}^{n-s} E_{t_{l_i}}),$$

then we have

$$E_r \oplus E_s \sim E_t.$$

And if there exists another bundle $E_{s'}$ such that

$$E_r \oplus E_{s'} \sim E_t.$$

Since E_r has a unique finite decomposition up to similarity, then we have

$$E_{s'} \sim \bigoplus_{i=0}^{n-s} E_{t_{l_i}} \sim E_s.$$

□

Before we give the proof of Theorem 4.7, we also need the following lemma from [17].

Lemma 4.12. ([17, Lemma 2.6]) *Let $\{P_1, \dots, P_m, P_{m+1}, \dots, P_N\}$ and $\{Q_1, \dots, Q_{m+1}, \dots, Q_N\}$ be two distinct unique decompositions of the vector bundle E_t . Suppose that for $1 \leq i \leq m$ and $w \in \Omega$,*

- (1) *there exists an $X_i \in GL(P_i \mathcal{H}, Q_i \mathcal{H})$ satisfying $X_i P_i t(w) = Q_i e(w)$ and that*
- (2) *there exists $Y \in GL(\mathcal{A}'(E_t))$ and a permutation $\Pi \in S_n$ satisfying $Y^{-1} P_i Y = Q_{\Pi(i)}$.*

Then for $r \in \{m+1, \dots, n\}$ and given Q_r , there exists $r' \in \{m+1, \dots, n\}$ and $Z_r \in GL(Q_r \mathcal{H}, P_{r'} \mathcal{H})$ satisfying $Z_r Q_r t(w) = P_{r'} t(w)$, $w \in \Omega$. Furthermore, if $r_1 \neq r_2$, then $r'_1 \neq r'_2$.

Proof of Theorem 4.7. Let $P \in P_n(\mathcal{A}'(E_t)) = P(\mathcal{A}'(E_{t^n}))$ be an idempotent. Then we have PE_{t^n} be a sub-bundle of E_{t^n} . Define map

$$\Gamma : V(\mathcal{A}'(E_t)) \rightarrow V^0(E_t)$$

with $\Gamma([p]_0) = \overline{PE_{t^n}}$.

Firstly, we will prove Γ is well defined. In fact, for any $P \sim Q \in [P]_0$, there exists positive integer n such that $P, Q \in \mathcal{A}'(E_{t^n})$. Since $Q = XPX^{-1}$, $X \in \mathcal{A}'(E_t)$, then we have

$$QE_{t^n} = XPX^{-1}E_{t^n} \sim PX^{-1}E_{t^n}.$$

And Note that $X, X^{-1} \in \mathcal{A}'(E_t)$, then we have

$$X^{-1}t^n(w) = t^n(w), \text{ for any } w \in \Omega.$$

Thus,

$$QE_{t^n} \sim PXE_{t^n},$$

and $\overline{QE_{t^n}} = \overline{PE_{t^n}}$. So Γ is well defined.

Secondly, we will prove that Γ is surjective. Suppose that E_r is a sub-bundle of E_{t^n} with dimension K , where n is positive integer. Suppose that

$$\mathcal{H}_r := \bigvee_{w \in \Omega} \{\gamma_1(w), \gamma_2(w), \dots, \gamma_K(w)\},$$

and P_r is the projection from \mathcal{H} to \mathcal{H}_r , then we have $P_r \in \mathcal{A}'(E_{t^n})$ and

$$P_r E_{t^n} \sim E_r.$$

Then it follows that Γ is surjective.

At last, we will prove that Γ is also injective. Let $P, Q \in \mathcal{A}'(E_{t^n})$. Suppose that there exists an invertible operator $X \in \mathcal{A}'(E_{t^n})$ such that

$$XPE_{t^n} = QE_{t^n}.$$

Let $\{p_1, p_2, \dots, p_m\}$ be a decomposition of P . Then $\{Xp_1X^{-1}, Xp_2X^{-1}, \dots, Xp_mX^{-1}\}$ be a decomposition of Q . In fact, we have

$$\begin{aligned} Xp_1X^{-1}QE_{t^n} + Xp_2X^{-1}QE_{t^n} + \dots + Xp_mX^{-1}QE_{t^n} &= Xp_1E_{t^n} + Xp_2E_{t^n} + \dots + Xp_mE_{t^n} \\ &= XPE_{t^n} \\ &= QE_{t^n}. \end{aligned}$$

Suppose that $\{p_{m+1}, p_{m+2}, \dots, p_N\}$ and $\{q_{m+1}, q_{m+2}, \dots, q_N\}$ be the decompositions of $(I - P)E_{t^n}$ and $(I - Q)E_{t^n}$ respectively. Then we have

$$\{p_1, p_2, \dots, p_N\} \text{ and } \{Xp_1X^{-1}, Xp_2X^{-1}, \dots, Xp_mX^{-1}, q_{m+1}, q_{m+2}, \dots, q_N\}$$

are two different decompositions of E_{t^n} . By the uniqueness of decomposition of E_{t^n} , there exists an invertible bounded linear operator $Y \in \mathcal{A}'(E_{t^n})$ such that $\{Y^{-1}P_iY\}$ is a rearrangement of $\{Xp_1X^{-1}, Xp_2X^{-1}, \dots, Xp_mX^{-1}\}, \{q_{m+1}, q_{m+2}, \dots, q_N\}$. By Lemma 4.12, for any $v \in \{m+1, m+2, \dots, n\}$, we can find $p_{v'}, v' \in \{m+1, \dots, n\}$ and $Z_v \in GL(L(q_v \mathcal{H}, p_{v'} \mathcal{H}))$ such that

$$Z_v q_v E_{t^n} = p_{v'} E_{t^n}, v'_1 = v'_2, \text{ when } v_1 = v_2.$$

Set $Z_k = X^{-1}|_{Xp_k X^{-1}\mathcal{H}}, k = 1, 2, \dots, m$, then we have that

$$Z = \sum_{k=1}^m Z_k + \sum_{v=m+1}^N Z_v \in GL\mathcal{A}'(E_{t^n}),$$

and

$$ZPZ^{-1} = Q.$$

It follows that Γ is injective. Since Γ is also a homomorphism, then we have

$$\text{Vect}^0(E_t) \cong \text{Vect}(\mathcal{A}'(E_{t^n}), K^0(E_t) \cong K_0(\mathcal{A}'(E_t))).$$

□

4.3. More general results on similarity. The precise relationship between the non-vanishing holomorphic sections of the atoms $t_i, 0 \leq i \leq n-1$, of a quasi-homogeneous holomorphic curve t and a holomorphic frame for t mandated in Definition 1.2 is at the heart of the proof of Theorem 3.1. We now push the limits of this definition a little and see if we can replicate some of our results. We begin by making the observation that starting with a quasi-homogeneous holomorphic curve t , we always have an operator T in the Cowen-Douglas class $B_n(\mathbb{D})$. This operator has an upper triangular decomposition as in Lemma 2.1. However, the other way round, starting with an operator T possessing such a decomposition, it may not be possible find a frame γ for the holomorphic Hermitian vector bundle E_T , which can be written as linear combinations of the non-vanishing sections of the atoms and their derivatives. In this section, we start with an operator T in the Cowen-Douglas class $B_n(\mathbb{D})$ assume that the operators appearing on the diagonal in its decomposition according to Theorem 1 are homogeneous operators in $B_1(\mathbb{D})$. Finally, we require that unlike quasi-homogeneous operators, there exists a holomorphic frame γ for E_T , which is a linear combination of the non-vanishing sections of the atoms and its derivatives as in Definition 1.2 except that the coefficients $\mu_{i,j}$ are allowed to be holomorphic functions rather than constants. For the remaining portion of this subsection, let $\mathcal{Q}_n(\mathbb{D})$ denote this class of operators, or for that matter, the corresponding holomorphic Hermitian vector bundles.

Proposition 4.13. *Let E_T and $E_{\tilde{T}}$ be two holomorphic Hermitian vector bundles in $\mathcal{Q}_n(\mathbb{D})$ with atoms $T_i, \tilde{T}_i, i = 0, 1$ an atomic decomposition $(S_{i,j})$ and $(\tilde{S}_{i,j})$, respectively. Suppose that $S_{i,i} = \tilde{S}_{i,i}, i = 0, 1$. Then E_T and $E_{\tilde{T}}$ are similar if and only if there exists an invertible holomorphic function $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $\tilde{S}_{0,1} = \phi(T_0)S_{0,1}$.*

For $i = 1, 2$, let \mathcal{H}_i be a Hilbert space of holomorphic function on \mathbb{D} possessing a reproducing kernel, say K_i , and T_i be the adjoint of the multiplication operator on \mathcal{H}_i . Assume that $\mathcal{H}_0 \subseteq \mathcal{H}_1$ and let $\iota : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ be the inclusion map. Then the adjoint ι^* of the inclusion map has the property $\iota^*(K_1(\cdot, w)) = K_0(\cdot, w), w \in \mathbb{D}$.

Lemma 4.14. *Assume that $K_i(z, w) = \frac{1}{(1-z\bar{w})^{\lambda_i}}, i = 0, 1$. Suppose that $S : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is a bounded linear operator with the intertwining property $T_0 S = S T_1$. Then there exists a holomorphic function ϕ such that $S = \phi(T_0)\iota^*$.*

Proof. The operators $T_i, i = 0, 1$ are in $B_1(\mathbb{D})$. If $S : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is a bounded linear operator and $T_0 S = S T_1$, then there exists a holomorphic function ψ such that $S^* = M_\psi$. This is easily proved as in [22, Section 5]. Let ϕ be the holomorphic function defined on the unit disc by the formula $\phi(\bar{w}) = \psi(w), w \in \mathbb{D}$. For any $f \in \mathcal{H}_0$, we have that

$$\begin{aligned} \langle f(z), \phi(T_0)\iota^*(K_1(z, w)) \rangle &= \langle f(z), \phi(\bar{w})K_0(z, w) \rangle \\ &= \overline{\phi(\bar{w})} \langle f(z), \phi(\bar{w})K_0(z, w) \rangle \\ &= \langle f(z), M_\psi^*(K_1(z, w)) \rangle \\ &= \langle f(z), S(K_1(z, w)) \rangle. \end{aligned}$$

Consequently, $S = \phi(T_0)\iota^*$.

□

Proof of Proposition 4.13. Let $T = \begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S}_{0,1} \\ 0 & T_1 \end{pmatrix}$ be the representations in upper-triangular operator matrices of E_t and $E_{\tilde{t}}$ respectively with the following properties:

By Lemma 2.1, we know that $E_t \sim E_{\tilde{t}}$ if and only if $T \sim \tilde{T}$. Then we only need to prove that T and \tilde{T} are similarity equivalent if and only if there exists invertible holomorphic function ϕ such that $\tilde{S}_{0,1} = \phi(T_0)S_{0,1}$.

To prove the necessity, note that there exists $\psi, \tilde{\psi} \in \mathcal{H}^\infty(\mathbb{D})$ such that

$$S_{0,1} = \psi(T_0)i^*, \tilde{S}_{0,1} = \tilde{\psi}(T_0)i^*$$

by Lemma 4.14. If there exists an invertible operator $Y = \begin{pmatrix} Y_{0,0} & Y_{0,1} \\ 0 & Y_{1,1} \end{pmatrix}$ such that

$$(4.1) \quad \begin{pmatrix} Y_{0,0} & Y_{0,1} \\ 0 & Y_{1,1} \end{pmatrix} \begin{pmatrix} T_0 & \psi(T_0)i^* \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & \tilde{\psi}(T_0)i^* \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} Y_{0,0} & Y_{0,1} \\ 0 & Y_{1,1} \end{pmatrix},$$

then $Y_{0,0}$ and $Y_{1,1}$ belong to the commutant of T_0 and T_1 , respectively. The operator Y is invertible and its inverse Y^{-1} is upper-triangular. The two operators $Y_{0,0}$ and $Y_{1,1}$ are also invertible. From Equation (4.1), we have that

$$Y_{0,0}\psi(T_0)i^* + Y_{0,1}T_1 = T_0Y_{0,1} + \tilde{\psi}(T_0)i^*Y_{1,1}.$$

As in the proof of the Lemma 3.2, we also have that

$$Y_{0,0}\psi(T_0)i^* = \tilde{\psi}(T_0)i^*Y_{1,1}.$$

Since $Y_{0,0}$ and $Y_{1,1}$ belongs to the commutant of T_0 and T_1 respectively, there exists invertible holomorphic functions $\phi_{0,0}$ and $\phi_{1,1} \in \mathcal{H}^\infty(\mathbb{D})$ such that

$$Y_{i,i} = \phi_{i,i}(T_i), i = 0, 1.$$

Consequently,

$$Y_{0,0}\psi(T_0)i^* - \tilde{\psi}(T_0)i^*Y_{1,1} = (\phi_{0,0}(T_0)\psi(T_0) - \phi_{1,1}(T_0)\tilde{\psi}(T_0))i^* = 0.$$

Thus $\tilde{\psi}(T_0) = \phi_{1,1}^{-1}(T_0)\phi_{0,0}(T_0)\psi(T_0)$. Set $\phi = \phi_{1,1}^{-1}\phi_{0,0}$, then we have that $\tilde{S}_{0,1} = \phi(T_0)S_{0,1}$ and $\phi(T_0)$ is invertible. This completes the proof of the necessary part.

We now prove the sufficiency. If there exists $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $\phi(T_0)$ is invertible $\tilde{S}_{0,1} = \phi(T_0)S_{0,1}$, then

$$\begin{pmatrix} \phi^{-1}(T_0) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T_0 & \tilde{S}_{0,1} \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} \phi^{-1}(T_0) & 0 \\ 0 & I \end{pmatrix}.$$

Therefore \tilde{T} is similar to T . □

The following proposition is similar to the one we have just proved for operators in $\mathcal{Q}_n(\mathbb{D})$, $n = 2$. Here we give the proof only for $n = 3$. The proof for an arbitrary n can be made up without involving any new ideas. It requires more of the same but somewhat tedious computations which we choose to skip.

Proposition 4.15. *Let $E_t, E_{\tilde{t}}$ be two holomorphic Hermitian vector bundles in $\mathcal{Q}_n(\mathbb{D})$ with atomic decompositions $(S_{i,j})$ and $(\tilde{S}_{i,j})$, respectively. Assume that $S_{i,i} = \tilde{S}_{i,i}$, $i = 0, 1, \dots, n-1$. If there exist holomorphic functions $\phi_{i,j} \in \mathcal{H}^\infty(\mathbb{D})$ such that $\tilde{S}_{i,j} = \phi_{i,j}(T_i)S_{i,j}$, $i, j = 0, 1, \dots, n-1, i < j$, such that $\tilde{S}_{i,j} = \phi_{i,j}(T_i)S_{i,j}$, $\phi_{i,j} \in \mathcal{H}^\infty(\mathbb{D})$, then E_t and $E_{\tilde{t}}$ are similarity equivalent if and only if $\phi_{i,j}(T_i)$ are all invertible and $\phi_{i,j} = \phi_{i,i+1}\phi_{i+1,i+2} \cdots \phi_{j-1,j}$.*

Proof. Let $T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}$, $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S}_{0,1} & \tilde{S}_{0,2} \\ 0 & T_1 & \tilde{S}_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}$ be the atomic decomposition of T and \tilde{T} , respectively. We prove that $T \sim \tilde{T}$ if and only if $\phi_{i,j}(T_i)$, $i, j \leq 2$ are invertible and $\phi_{0,2} = \phi_{0,1}\phi_{1,2}$. Since $T \sim \tilde{T}$, by Lemma 2.6, there exists an invertible operator $X = \langle\langle X_{i,j} \rangle\rangle$, which is upper triangular and such that

$$\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_0 & \tilde{S}_{0,1} & \tilde{S}_{0,2} \\ 0 & T_1 & \tilde{S}_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix}.$$

Then we have that

$$\begin{pmatrix} X_{0,0} & X_{0,1} \\ 0 & X_{1,1} \end{pmatrix} \begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & \tilde{S}_{0,1} \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} \\ 0 & X_{1,1} \end{pmatrix}$$

and

$$\begin{pmatrix} X_{1,1} & X_{1,2} \\ 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} T_1 & S_{1,2} \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_1 & \tilde{S}_{1,2} \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} X_{1,1} & X_{1,2} \\ 0 & X_{2,2} \end{pmatrix}.$$

The inverse X^{-1} is also upper-triangular. Now, both $\begin{pmatrix} X_{0,0} & X_{0,1} \\ 0 & X_{1,1} \end{pmatrix}$ and $\begin{pmatrix} X_{1,1} & X_{1,2} \\ 0 & X_{2,2} \end{pmatrix}$ are invertible and consequently, by Proposition 4.13, we see that $\phi_{0,1}$ and $\phi_{1,2}$ must be invertible.

Set $\phi_0 = 1$, $\phi_1 = \phi_{0,1}$, $\phi_2 = \phi_{0,1}\phi_{1,2}$, $X_i := X_{i,i} = \phi_i(T_i)$, $\phi_i \in \mathcal{H}^\infty(\mathbb{D})$. We have that

$$X\tilde{T}X^{-1} = \begin{pmatrix} X_0T_0X_0^{-1} & X_0\tilde{S}_{0,1}X_1^{-1} & X_0\tilde{S}_{0,2}X_2^{-1} \\ 0 & X_1T_1X_1^{-1} & X_1\tilde{S}_{1,2}X_2^{-1} \\ 0 & 0 & X_2T_2X_2^{-1} \end{pmatrix}.$$

Consequently,

$$\begin{aligned} \tilde{T} &\sim \begin{pmatrix} X_0T_0X_0^{-1} & X_0\tilde{S}_{0,1}X_1^{-1} & X_0\tilde{S}_{0,2}X_2^{-1} \\ 0 & X_1T_1X_1^{-1} & X_1\tilde{S}_{1,2}X_2^{-1} \\ 0 & 0 & X_2T_2X_2^{-1} \end{pmatrix} \\ &= \begin{pmatrix} X_0T_0X_0^{-1} & X_0\phi_{0,1}(T_0)S_{0,1}X_1^{-1} & X_0\phi_{0,2}(T_0)S_{0,2}X_2^{-1} \\ 0 & X_1T_1X_1^{-1} & X_1\phi_{1,2}(T_1)S_{1,2}X_2^{-1} \\ 0 & 0 & X_2T_2X_2^{-1} \end{pmatrix} \\ &= \begin{pmatrix} T_0 & S_{0,1} & X_0\phi_{0,2}(T_0)S_{0,2}X_2^{-1} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} \\ &= \begin{pmatrix} T_0 & S_{0,1} & \phi_{0,2}(T_0)S_{0,2}\phi_2^{-1}(T_2) \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} \end{aligned}$$

Now set

$$\bar{T} = \begin{pmatrix} T_0 & S_{0,1} & \phi_{0,2}(T_0)S_{0,2}\phi_2^{-1}(T_2) \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix},$$

and $\bar{S}_{0,2} = \phi_{0,2}(T_0)S_{0,2}\phi_2^{-1}(T_2)$. Since $T \sim \tilde{T} \sim \bar{T}$, by Lemma 3.5, we find an invertible operator $X = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix}$ such that

$$\begin{pmatrix} T_0 & S_{0,1} & S_{0,2} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} T_0 & S_{0,1} & \bar{S}_{0,2} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}.$$

Then we have

$$(4.2) \quad X_{0,0}\bar{S}_{0,2} - S_{0,2}X_{2,2} = S_{0,0}X_{0,2} - X_{0,2}S_{2,2},$$

and $X_{i,i} \in \mathcal{A}'(T_i)$. In the following, we will describe $X_{0,0}\bar{S}_{0,2} - S_{0,2}X_{2,2}$.

Note that $X_{i,i}(t_i)(w) = \phi(w)t_i(w)$, $w \in \Omega$, where ϕ is a holomorphic function on \mathbb{D} and

$$\begin{aligned}
 (4.3) \quad X_{0,0}\bar{S}_{0,2} - S_{0,2}X_{2,2}(t_2) &= X_{0,0}(\phi_{0,2}(T_2)S_{0,2}(\phi_2^{-1}(T_2)(t_2))) - S_{0,2}(\phi t_2) \\
 &= -X_{0,0}(\phi_{0,2}(T_0)\phi_2^{-1}t_0^{(1)}) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}X_{0,0}(\phi_{0,2}(T_0)t_0^{(1)}) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}X_{0,0}(\phi_{0,2}t_0^{(1)} + \phi_{0,2}^{(1)}t_0) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}X_{0,0}(\phi_{0,2}t_0^{(1)}) - \phi_2^{-1}\phi_{0,2}^{(1)}X_{0,0}(t_0) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}\phi_{0,2}X_{0,0}(t_0^{(1)}) - \phi_2^{-1}\phi_{0,2}^{(1)}(\phi t_0) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}\phi_{0,2}((\phi t_0)^{(1)}) - \phi_2^{-1}\phi_{0,2}^{(1)}(\phi t_0) + \phi t_0^{(1)} \\
 &= -\phi_2^{-1}\phi_{0,2}(\phi t_0^{(1)}) - \phi_2^{-1}\phi_{0,2}(\phi^{(1)}t_0) - \phi_2^{-1}\phi_{0,2}^{(1)}(\phi t_0) + \phi t_0^{(1)} \\
 &= \phi(1 - \phi_2^{-1}\phi_{0,2})t_0^{(1)} - \phi_2^{-1}\phi_{0,2}\phi^{(1)}t_0 - \phi_2^{-1}\phi_{0,2}^{(1)}\phi t_0 \\
 &= \phi(-1 + \phi_2^{-1}\phi_{0,2})S_{0,2}(t_2) - S_{0,1}S_{1,2}(\phi_2^{-1}(\phi_{0,2}\phi)^{(1)})S_{2,2}(t_2).
 \end{aligned}$$

Let $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_0)$ satisfies that $S(t_2) = \phi(\phi_2^{-1}\phi_{0,2} - 1)S_{0,2}(t_2)$. Let $\psi = \phi(\phi_2^{-1}\phi_{0,2} - 1)$, then we have $S(t_2) = \psi(S_{0,0})S_{0,2}(t_2)$. Note that $S_{0,1}S_{1,2}(\phi_2^{-1}(\phi_{0,2}\phi)^{(1)})S_{2,2} \in \text{ran } \sigma_{T_0, T_2}$. By (4.2) and (4.3), there exist linear bounded operator Z such that

$$S(t_2) = (T_0Z - ZT_2)(t_2).$$

That means

$$\begin{aligned}
 S_{0,2}(t_2) &= T_0Z(\psi^{-1}(T_2)(t_2)) - ZT_2(\psi^{-1}(T_2)(t_2)) \\
 &= T_0Z\psi^{-1}(T_2)(t_2) - Z\psi^{-1}(T_2)T_2(t_2).
 \end{aligned}$$

This is a contradiction to the fact $S_{0,2} \notin \text{ran } \sigma_{T_0, T_2}$. So we have that $\phi_2 = \phi_{0,2} = \phi_{0,1}\phi_{1,2}$.

To prove the proposition the other way round, assume that $\tilde{T} = \begin{pmatrix} T_0 & \phi_{0,1}(T_0)S_{0,1} & \phi_{0,2}(T_0)S_{0,2} \\ 0 & T_1 & \phi_{1,2}(T_0)S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}$

and $\phi_{0,1}$, $\phi_{1,2}$ and $\phi_{0,2} = \phi_{0,1}\phi_{1,2}$ are all invertible. Then we have

$$\tilde{T} \sim \begin{pmatrix} T_0 & S_{0,1} & \phi_{0,2}(T_0)S_{0,2}(\phi_{0,1}\phi_{1,2})^{-1}(T_2) \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix} = \begin{pmatrix} T_0 & S_{0,1} & \phi_{0,2}(T_0)S_{0,2}\phi_{0,2}^{-1}(T_2) \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}.$$

We also have

$$\begin{aligned}
 &\begin{pmatrix} \phi_{0,2}^{-1}(T_0) & 0 & 0 \\ 0 & \phi_{0,2}^{-1}(T_1) & 0 \\ 0 & 0 & \phi_{0,2}^{-1}(T_2) \end{pmatrix} \begin{pmatrix} T_0 & S_{0,1} & \phi_{0,2}(T_0)S_{0,2}\phi_{0,2}^{-1}(T_2) \\ T_1 & S_{1,2} & \\ & T_2 & \end{pmatrix} \begin{pmatrix} \phi_{0,2}(T_0) & 0 & 0 \\ 0 & \phi_{0,2}(T_1) & 0 \\ 0 & 0 & \phi_{0,2}(T_2) \end{pmatrix} \\
 &= \begin{pmatrix} T_0 & \phi_{0,2}^{-1}(T_0)S_{0,1}\phi_{0,2}(T_1) & \phi_{0,2}^{-1}(T_0)\phi_{0,2}(T_0)S_{0,2}\phi_{0,2}^{-1}(T_2)\phi_{0,2}(T_2) \\ 0 & T_1 & \phi_{0,2}^{-1}(T_1)S_{1,2}\phi_{0,2}(T_2) \\ 0 & 0 & T_2 \end{pmatrix} \\
 &= \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} \\ 0 & T_1 & S_{1,2} \\ 0 & 0 & T_2 \end{pmatrix}.
 \end{aligned}$$

□

4.4. The Halmos' question. The well-known question of Halmos asks if $\varrho : \mathbb{C}[z] \rightarrow \mathcal{L}(\mathcal{H})$ is a continuous (for $p \in \mathbb{C}[z]$, the norm $\|p\| = \sup_{z \in \mathbb{D}} |p(z)|$) algebra homomorphism induced by an operator S , that is, $\varrho(p) = p(S)$, then does there exist an invertible linear operator L and a contraction T on the Hilbert space \mathcal{H} so that $S = LTL^{-1}$. After the question was raised in [10, Problem 6], an affirmative answer for several classes of operators were given. A counter example was found by Pisier in 1996 (cf. [27]).

It was pointed out in a recent paper of the third author with Korányi [23] that the Halmos' question has an affirmative answer for homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$. This was based on the description of equivalence classes of homogeneous operators under invertible bounded linear transformations. In the terminology of this paper, (multiplicity free) homogeneous operators are irreducible and also strongly reducible. Now, we have this for quasi-homogeneous operators, see Theorem 3.4. Thus it is natural to ask if the Halmos' question has an affirmative answer for quasi-homogeneous operators. If $\Lambda(t) \geq 2$, the answer is evidently "yes":

In this case, the quasi-homogeneous operator T is similar to the n -fold direct sum of the homogeneous operators T_i (adjoint of the multiplication operator) acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$, $i = 0, 1, \dots, n-1$. Now, if $\lambda_0 \geq 1$, this direct sum is contractive and we are done. If $\lambda_0 < 1$, then T_0 is not even power bounded and therefore neither is the operator T . So, there is nothing to prove when $\lambda_0 < 1$.

If $\Lambda(t) < 2$, then the operator T is strongly irreducible. Therefore, we can't answer the Halmos' question purely in terms of the atoms of the operator T . Never the less, the answer is affirmative even in this case. To show this, we need a preparatory lemma.

Lemma 4.16. *Suppose that t is a quasi-homogeneous holomorphic curve. Assume that $\Lambda(t) < 2$ and $\lambda_0 \geq 1$. Then the operator T is not power bounded.*

Proof. The top 2×2 block in the atomic decomposition of the quasi-homogeneous operator T is of the form $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$. As always, we assume that the operators T_0 and T_1 are the adjoints of the multiplication operator on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$, respectively. The operator $S_{0,1}$ has the intertwining property $T_0 S_{0,1} = S_{0,1} T_1$.

Let ι denote the inclusion map from $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ to $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$. Then $\iota^*(t_1)(w) = t_0(w)$, $w \in \mathbb{D}$, and the operator $S_{0,1}$ must be of the form $\phi(T_0)\iota^*$ for some holomorphic function ϕ on the unit disc \mathbb{D} , as we have shown in Lemma 4.14. Indeed, $S_{0,1}(t_1(w)) = \phi(w)t_1(w) = \phi(T_0)\iota^*(t_1(w))$.

Without loss of generality, we assume that $\phi(w) = \sum_{i=0}^{\infty} \phi_i w^i$ and $\phi_0 \neq 0$. For $j = 0, 1$, the set of vectors $e_\ell^{(\lambda_j)} := \sqrt{a_\ell(\lambda_j)} z^\ell$, $\ell \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$. Then we have that

$$T_0^{n-1}(e_\ell(\lambda_0)) = \prod_{i=\ell-n+1}^{\ell-1} w_i(\lambda_0) e_{\ell-n+1}(\lambda_0), S_{0,1}(e_\ell(\lambda_1)) = \phi_0 \frac{\prod_{i=0}^{\ell-1} w_i(\lambda_1)}{\prod_{i=0}^{\ell-1} w_i(\lambda_0)} e_\ell(\lambda_0).$$

Consequently,

$$nT_0^{n-1}S_{0,1}(e_\ell(\lambda_1)) = n\phi_0 \frac{\prod_{i=0}^{\ell-1} w_i(\lambda_1)}{\prod_{i=0}^{\ell-n} w_i(\lambda_0)} e_{\ell-n+1}(\lambda_0)$$

Since $w_i(\lambda_0) = \sqrt{\frac{i+1}{i+\lambda_0}}$ and $w_i(\lambda_1) = \sqrt{\frac{i+1}{i+\lambda_1}}$, it follows that

$$\prod_{i=0}^{\ell-1} w_i(\lambda_1) \sim ((\ell-1)^{\frac{1-\lambda_1}{2}}) \text{ and } \prod_{i=0}^{\ell-n} w_i(\lambda_0) \sim ((\ell-n)^{\frac{1-\lambda_0}{2}})$$

implying

$$\frac{\prod_{i=0}^{\ell-n} w_i(\lambda_1)}{\prod_{i=0}^{\ell-1} w_i(\lambda_0)} \sim \left(\frac{(\ell-n)^{\frac{\lambda_0-1}{2}}}{(\ell-1)^{\frac{\lambda_1-1}{2}}} \right).$$

If we choose $\ell = 2n + 1$, then we have

$$\frac{(\ell - n)^{\frac{\lambda_0 - 1}{2}}}{(\ell - 1)^{\frac{\lambda_1 - 1}{2}}} \sim \left(\frac{1}{n^{\frac{\lambda_1 - \lambda_0}{2}}} \right) \text{ for large } n.$$

Hence $\|nT_0^{n-1}S_{0,1}\| \rightarrow \infty$ as $n \rightarrow \infty$.

Let $T_{|_{2 \times 2}}$ denote the top 2×2 block $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$ in the operator T . Since $T_{|_{2 \times 2}}^n = \begin{pmatrix} T_0^n & nT_0^{n-1}S_{0,1} \\ 0 & T_1^n \end{pmatrix}$, and $\|T_{|_{2 \times 2}}^n\| \geq \|nT_0^{n-1}S_{0,1}\|$, it follows that $\|T_{|_{2 \times 2}}^n\| \rightarrow \infty$ as $n \rightarrow \infty$. Clearly, $\|T^n\| \geq \|T_{|_{2 \times 2}}^n\|$ completing the proof. \square

Since a quasi-homogeneous operator for which $\lambda_0 < 1$ can't be power bounded, the lemma we have just proved shows that if T is quasi-homogeneous and $\Lambda(t) < 2$, then the operator T is not power bounded. Therefore we have proved the following theorem answering the Halmos' question in the affirmative.

Theorem 4.17. *If a quasi-homogeneous operator T has the property $\|p(T)\|_{\text{op}} \leq K\|p\|_{\infty, \mathbb{D}}$, $p \in \mathbb{C}[z]$, then it must be similar to a contraction.*

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